James Amoss

Markham

LOVELL'S SERIES OF SCHOOL BOOKS.

ELEMENTS

OF

ALGEBRA;

DESIGNED FOR THE USE OF

CANADIAN GRAMMAR AND COMMON SCHOOLS.

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ERRATA.

Page	58	last line	for	$x + 3x - 4$	read	$x^2 + 3x - 4$.
"	106	Quest. 18	"	$- 0$	"	$= 0$.
"	123	" 1	"	$49c$	"	$49c^2$
"	153	" 7	"	$x + 2$	"	$x + 1$
"	165	" 12	"	$+ 2x^2$	"	$- 2x^2$
"	170	" 36	"	$(4x^2$	"	$(4x^2 + 3)^{= a}$
"	197	" 1	"	$x \propto y$	"	$x \propto \sqrt{y}$
"	251	" 21	"	3	"	9
"	251	" 24	"	pail with	"	paid with 100 pieces in
"	256	" 58	"	ninth	"	fourth

PREFACE.

The following Treatise is respectfully submitted by the author to the teachers of Canada, in the confident belief that it will materially lighten the labor of the instructor, and, at the same time, facilitate the pupil's progress and his thorough comprehension of the principles of the science of algebra. It is the earnest hope of the author that it may meet with the same flattering reception, and very general introduction into the schools of the country, that his fellow-teachers have so kindly accorded to his previous productions.

The order of succession of the different chapters depends of course mainly on their importance and difficulty, and that here adopted is the one that appears preferable to the author; but, as every chapter is nearly independent of the others, the teacher can easily modify the arrangement to suit himself.

The aim of the work is to embrace all that can be profitably discussed in the time usually allotted to a common and grammar school course; and, indeed, this volume will be found to contain at least as much of the subject, as is required to be read for the ordinary degree of B. A. in the British and Canadian Universities. Chapters on continued fractions, logarithmic series, probabilities, and

the general theory of equations were prepared, but, in accordance with the advice of some of the leading educators of the province, they were omitted as unsuited to the design of the work, and to the requirements of common or grammar schools.

The author has approached the subject with the conviction, founded on many years' experience as a teacher of mathematics, that the science of algebra tries, beyond all others, the powers and patience of the learner. The pupil is commonly introduced to it while his mind is yet in an undeveloped state; its language is new to him, and he is unprepared by previous training to comprehend its abstractions. The difficulties which thus beset his path are, of course, for the most part, only to be overcome by his own perseverance, aided by the knowledge and ingenuity of his instructor, yet it appears to the author that very much also depends upon the style and thoroughness and adaptation of the text-book employed. Accordingly in the preparation of this volume no pains have been spared in rendering the statement of principles, and the demonstration of theorems as clear and concise as possible, or in fully illustrating each rule by numerous examples carefully worked out and explained, or in selecting and arranging the examples of an exercise so as to begin with the simple, and gradually pass on to the more difficult.

The author hopes that while he has insisted upon great thoroughness by numerous and appropriate problems, he has, at the same time, rendered the pupil's advancement easy and certain by the many explanations and illustrations introduced.

The great majority of the problems and exercises are new,—being now published for the first time, but there are

also a number already familiar to the teacher. In selecting these the author has, he believes, in every case rigidly adhered to the rule, adopted by Todhunter, Colenso, and others, of not inserting a problem unless it had already appeared in at least two British authors—in which case it is to be regarded as common property.

Recognizing the fact that very many of the pupils of our common and grammar schools study with the view of completing their education at some one of our excellent Canadian universities, the author has, at the end of the book, introduced a collection of problems and theorems, embracing among others all or nearly all of the pass and honor work in algebra which has been given on the examination papers of the university of Toronto during the last eight or ten years. These will serve to shew the pupil the style of questions he is expected to answer at our universities, and will, at the same time, in a measure prepare him for his examinations.

As no teacher would think of introducing his pupils to arithmetic without, to some extent at least, first drilling them in notation and numeration, so no intelligent teacher will neglect to drill his pupils in *algebraic notation and numeration* before introducing them to the ordinary rules. The teacher is respectfully referred to exercises ii, iii, and iv, and is recommended to extend and continue these until his pupil is thoroughly and practically acquainted with the definitions.

Well knowing the great inconvenience to both teacher and pupils of inaccuracies and mistakes in a work on algebra, the author has subjected this treatise to a searching revision; and he believes that the few corrections marked on the back of the title page are the only errors in the

letter-press of the exercises and answers of the work The teacher is respectfully recommended to cause his pupils to make the six or eight trifling alterations there indicated in the body of the work with pen and ink.

A key, containing full solutions to all the more difficult problems, is in press and will be issued almost immediately.

TORONTO, January, 1864.

CONTENTS.

ALGEBRA.

SECTION I.

DEFINITIONS AND AXIOMS.

1. Algebra is Arithmetic *generalized;* or, in other words, it is a kind of Arithmetic in which the *numbers* or *quantities* under consideration are *represented by letters,* and the *operations* to be performed on these *indicated by signs.*

2. The symbols employed in Algebra are of five kinds viz.:—

1st. Symbols of *Quantity.*
2nd. Symbols of *Operation.*
3rd. Symbols of *Relation.*
4th. Symbols of *Aggregation.*
5th. Symbols of *Deduction.*

SYMBOLS OF QUANTITY.

3. The symbols of quantity are the Arabic numerals and the letters of the alphabet.

4. Algebraic quantities are of two kinds, viz.:—

1st. *Known* or *determined quantities*, or those which may be assumed to be of any value whatever.

2nd. *Unknown* or *undetermined quantities*, or those whose value can be determined only by actually performing the operations involved in the solution of the problem, &c.

5. The first letters of the alphabet, a, b, c, d, &c., are used to represent *known* quantities, and the last letters of the alphabet, x, y, z, w, v, &c., are employed to represent *unknown* quantities.

6. The symbol 0 is called zero, and indicates the absence of quantity, or it represents a quantity *infinitely small*, i.e. less than any assignable quantity.

7. The symbol ∞ is called *infinity*, and denotes a quantity *infinitely great*, i.e. greater than any assignable quantity.

NOTE.—The symbol $\propto$ is also employed to indicate that one quantity varies as another. [See the section on Variation.]

SYMBOLS OF OPERATION.

8. The symbols of operation are $+$, $-$, $\sim$, $\times$, $\div$, $^{2, 3, 4, \&c.,}$ $^{\frac{1}{2}}$, $^{\frac{1}{3}}$, $^{\frac{1}{4}}$, $^{\&c.,}$ $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, &c.

9. The sign $+$ is called *plus* or *the sign of addition*, and indicates that the quantities between which it is written are to be added together.

Thus, $7 + 9$, read 7 plus 9, means that 7 and 9 are to be added together.

$a + b$, read a plus b, denotes that a and b are to be added together.

10. The sign $-$ is called *minus* or *the sign of subtraction*, and indicates the subtraction of the quantity following it from the quantity preceding it.

Thus, $11-6$, read 11 *minus* 6, means that 6 is to be taken from 11.

$a-b$, read a minus b, implies that the quantity a has to be decreased by the quantity b.

11. The multiplication of one algebraic quantity by another may be indicated—

1st. By writing the sign $\times$ between them.
2nd. By writing a dot . between them.
3rd. By writing them in juxtaposition.

Thus, $a \times b$ and $a \,.\, b$ and ab each indicate the multiplication of the quantity a by the quantity b, and are read a multiplied into b, or simply a into b. The last is the method commonly employed to indicate multiplication in algebra. Arithmetical multiplication is expressed only by the sign $\times$, the other methods being obviously inapplicable to numbers.

NOTE.—Quantities connected by the sign $+$ or $\times$ may be read in any order. Thus $6+3$ is the same in value as $3+6$, for each is equal to 9; so 6×5 is the same in value as 5×6, for each is equal to 30.

12. There are three modes of representing the division of one quantity by another, namely, by writing between them the common arithmetical sign of division $\div$ or by writing between them either the sign : or the sign —

Thus, $a \div b$, and $a:b$, and $\frac{a}{b}$ each represent the division of the quantity a by the quantity b. The last method, i.e. writing the quantities in a fractional form is that usually made use of in algebra.

NOTE.—Quantities connected by the sign $-$ or $\div$ must be read just as they are written. Thus $8-3$ is very different in value from $3-8$; so $12 \div 4$ is quite distinct from $4 \div 12$.

13. The symbol $\sim$ written between two quantities indicates that the less is to be subtracted from the greater.

Thus, $7 \sim 3$ or $3 \sim 7$, read the difference between 3 and 7, denotes that 3 is to be taken from 7. So $a \sim b$ or $b \sim a$ indicates that a is to be taken from b or b from a, according as a is less or greater than b.

NOTE.—The symbol $\sim$ is employed only when it is not known which of the two quantities is the greater.

14. An *exponent* is a small figure or letter placed to the right of a quantity to show *how often* it is taken as a *factor*.

Thus, $a^3 = aaa$, the 3 indicating that a is to be taken three times as factor.

$m^7 = mmmmmmm$, the 7 showing that m is to be taken seven times as factor.

$(a+b)^n = (a+b)\ (a+b)\ (a+b)$, &c., to n terms, the n denoting that the quantity $(a+b)$ is to be taken as factor as many times as there are units in n.

NOTE.—When the exponent is *unity*, it is not commonly expressed.

15. The extraction of a root is indicated either by writing it with a fractional index or by placing it under the *radical sign* $\sqrt{\ }$.

Thus, $\sqrt{7}$ or $7^{\frac{1}{2}}$ denotes the *square root* of 7.

$\sqrt[3]{a}$ or $a^{\frac{1}{3}}$ denotes the *cube root* of a.

$\sqrt[n]{a}$ or $a^{\frac{1}{n}}$ denotes the n^{th} *root* of a, &c.

16. The number 3, or 4, or 5, &c., placed in the radical sign or as denominator in the fractional exponent, is called the *index of the root*. The index 2 is never used in connection with the radical sign; thus, $\sqrt{a}$ is the same as $\sqrt[2]{a}$.

17. When a fractional exponent is employed the numerator denotes the *power* and the denominator the *root* to be taken.

Thus, $a^{\frac{4}{7}}$ denotes the 4th power of the 7th root of a or the 7th root of the 4th power of a.

$x^{\frac{m}{n}}$ indicates the nth root of the mth power of x, or the mth power of the nth root of x.

SYMBOLS OF RELATION.

18. The symbols of relation are $:$, $=$, $::$, $>$, and $<$.

19. The symbol $:$ denotes *ratio*.

Thus, $a:b$ denotes the ratio of a to b.

20. The symbol $=$ is the sign of equality.

Thus, $7+4=5+6$ denotes that the sum of 7 and 4 is equal to the sum of 5 and 6. $a=b$ denotes that a is equal in value to b.

21. The symbol $::$ is also a sign of equality, but is used only to denote the *equality of ratios*.

Thus $9:27::5:15$ denotes that the ratio of 9 to 27 is equal to that of 5 to 15.

$a:b::c:d$ denotes that the ratio of a to b is equal to that of c to d.

22. The symbol $>$ *greater than*, and the symbol $<$ *less than*, are signs of inequality.

Thus $7>5$ denotes that 7 is greater than 5.
$a>b$ denotes that a is greater than b.
$5<7$ denotes that 5 is less than 7.
$a<b$ denotes that a is less than b.

NOTE.—The opening of the angle is always towards the greater quantity.

SYMBOLS OF AGGREGATION.

23. The symbols of aggregation are —, |, ( ), { }, and [ ].

24. The symbol — is called a *vinculum*, and indicates that the quantities over which it is placed are to be regarded as constituting *but one quantity*.

Thus, $\overline{a + b - c} \times d$ means that the quantity formed by the subtraction of c from the sum of a and b is to be multiplied by d.

$\sqrt{m + x + y}$ denotes that the square root of the sum of m, x, and y is to be taken.

25. The symbol | is called *a bar*, and indicates that the quantities in the column directly preceding it are to be considered as forming but one quantity.

Thus, $\left.\begin{matrix} +a \\ +b \\ -c \end{matrix}\right|^2$ denotes that the quantity formed by the subtraction of c from the sum of a and b is to be *squared*.

26. The *parentheses* ( ), *braces* { }, and *brackets* [ ], denote that the quantities contained within them are to be regarded as constituting one quantity.

Thus $(a + b)x$ denotes that the sum of a and b is to be multiplied by x.

$\{a - (b + c)\}^3$ indicates that the sum of b and c is to be taken from a and the remainder cubed.

$[a - \{m - (b + c)\,x\}]y$ denotes that $(b + c)x$ is to be taken from m and the remainder subtracted from a, and that this final remainder is to be multiplied by y.

SYMBOLS OF DEDUCTION.

27. The symbols of deduction are $\therefore$ and $\because$

28. The symbol $\therefore$ is equivalent to *therefore*, *whence*, *thence*, *consequently*, *from which we infer*, &c.

Thus, $a = b$ and $c = b \therefore a = c$.

29. The symbol $\because$ signifies *since* or *because*.

Thus, $a = c \because a = b$ and $c = b$.

30. The parts of an algebraic expression separated from each other by the sign of addition or subtraction, expressed or understood, are called *terms*.

Thus, a is an algebraic expression of one term and is called a *monomial*.

$a+b$ is an algebraic expression of two terms, and is called a *binomial*.

$a+b-c$ is an algebraic expression of three terms, and is called a *trinomial*.

$2a+3b-4c+x-y$ is an algebraic expression of five terms, and is called a *multinomial* or *polynomial*.

31. The parts of an algebraic expression connected by the sign of multiplication, expressed or understood, are called *factors*.

Thus, the factors of the expression ab are a and b.

The factors of the expression a^2bc^3 are a, a, b, c, c, and c.

The factors of the expression $(x-y)^2(a-my)^3$ are $(x-y)$, $(x-y)$, $(a-my)$, $(a-my)$, and $(a-my)$.

32. The terms of an algebraic expression which are preceded by the sign $+$ are called *additive* or *positive terms*; those preceded by the sign $-$ are called *subtractive* or *negative terms*.

Thus, in the expression $7a-3c-4d+5m+7x+8y-mx-ab$, the terms $7a$, $5m$, $7x$, and $8y$ are additive or positive, and the terms $3c$, $4d$, mx, and ab are subtractive or negative.

NOTE.—When no sign is expressed before a quantity it is understood to be additive. Thus, in the above expression, $7a$ is written for $+7a$.

33. A *coefficient* is a number or letter written to the left of a quantity to show how often it is to be taken as *addend*.

Thus, $7a$ indicates that the sum of seven a's is to be taken in an additive sense.

$-5x$ denotes that the sum of five $-x$'s is to be taken in an additive sense.

Here 7 is called the coefficient of a, 5 the coefficient of x, &c.

34. *Like algebraic quantities* are those that consist of the same letters affected by the same exponents.

Thus, $-3a$, $-2a$, $4a$, $-5a$ are like quantities.

a^2bc, $7a^2bc$, $-3a^2bc$ are like quantities.

$5(a^2-b+c^4)$, $7(a^2-b+c^4)$ and $\frac{1}{16}(a^2-b+c^3)$ are like quantities.

But a^2bc, and ab^2c are unlike quantities, because the same letter is not affected by the exponent 2.

So also $a^2b^3c^4$, $a^3b^2c^4$, and $a^4b^3c^3$ are unlike quantities.

35. *Homogeneous terms* are those in which the sum of the exponents of the literal factors in each are equal.

Thus $2a^4y$ and $7a^2y^3$ are homogeneous, and the sum of the exponents of the literal factors in each being 5, they are called homogeneous terms of *five dimensions.*

$3ax^3y^3$, $4a\ x^2y^3$, $9a^6y$, $7axy^5$, and y^7 are homogeneous, the sum of the exponents of the literal factors in each term being 7, and they are called homogeneous terms of *seven dimensions.*

36. The *reciprocal* of a quantity is unity divided by that quantity.

Thus, the reciprocal of 3 is $\frac{1}{3}$, of a is $\frac{1}{a}$, of $\frac{b}{m}$ is $\frac{m}{b}$, of $\frac{2}{7}$ is $\frac{7}{2}$, &c.

AXIOMS.

37. An *axiom* is a theorem which cannot be reduced to a simpler theorem.

The following are the principal axioms made use of in algebra:—

I. *The whole is equal to the sum of all its parts.*

II. *If equal quantities or the same quantity be added to equal quantities, the sums will be equal.*

III. *If equal quantities or the same quantity be subtracted from equal quantities, the remainders will be equal.*

IV. *If equals be multiplied by equals or by the same, the products will be equal.*

V. *If equals be divided by equals or by the same, the quotients will be equal.*

VI. *If the same quantity be both added to and subtracted from another, the latter will not be altered in value.*

VII. *If equals or the same be added to or subtracted from unequal quantities, the sums or remainders will be unequal.*

VIII. *If unequals be multiplied or divided by equals or by the same, the products or the quotients will be unequal.*

IX. *Equimultiples of the same quantities or of equal quantities are equal to one another.*

X. *Equal powers or equal roots of the same or of equal quantities are equal to one another.*

XI. *Things which are equal to the same thing are equal to one another.*

EXERCISE I.

1. What is algebra? (1)
2. Classify algebraic symbols. (2)
3. What are the symbols of quantity? (3)
4. What are the symbols of operation? (8)
5. Write down the symbols of relation. (18)
6. Express the symbols of aggregation. (23)
7. What are the symbols of deduction? (27)
8. What letters are employed to denote known quantities? Unknown quantities? (5)
9. What is the meaning of the symbol 0? Of the symbol ∞? (6 and 7)
10. What is an exponent? (14)
11. What is a coefficient? (33)
12. What are the terms of an algebraic expression? (30)
13. What are the factors of an algebraic expression? (31)
14. What is a monomial? A binomial? A multinomial? (30)
15. What are like quantities? (34)
16. What are homogeneous terms? (35)
17. What are additive terms? (32)
18. What are subtractive terms? (32)

19. What are positive and negative terms? (32)

20. When no sign is expressed before a term how is it regarded? (32)

21. How many ways have we of indicating the extraction of a root? (15)

22. What is the index of the root? (16)

23. What does the denominator of a fractional index denote? What the numerator? (17)

24. How are quantities connected by the sign $+$ or $\times$ to be read? How those connected by the sign $-$ or $\div$? (11 & 12, Notes)

25. What are axioms? (37)

26. Give the principal axioms employed in algebra. (37)

EXERCISE II.

Read the following expressions and explain what each indicates .—

1. a, $5a$, $9c^2$, $4a^{\frac{1}{2}}$, $x^{\frac{4}{5}}$, $\frac{3}{7}(a+b)$, $5x(y+z-c)$, $\left.\begin{array}{r}+c\\-3m\\4x\end{array}\right| a$

2. $3a+4-7c$, $(x-y-z)^3$, abc, $\frac{my}{xz}$, $\sqrt[5]{ab\ (m+xy^4)}$

3. $(m+x) \sim (x+y)$, $a^{\frac{m}{n}}$, $a^2-b^2=(a+b)\ (a-b)$, $\frac{a^2+2ab-x^3}{3a-4c^{\frac{1}{2}}+\sqrt[3]{m}}$

4. $7+a>a-3$, $a^{\frac{1}{3}}<a^{\frac{1}{2}}$, $\{a-(b+c)\}^{\frac{2}{5}}=\sqrt[5]{(a-b-c)^2}$

5. $\because a>b$ and $b>c \therefore c<a$.

6. $a-3ab+4a^2c^2-7abx+3y^2-7\sqrt{2y}+(a-b+c)^{\frac{2}{3}}-\sqrt[4]{xy}+(a\sim m)$.

Of the above algebraic expressions :—

7. Which are monomials?
8. Which are binomials?
9. Which are multinomials?
10. Which are coefficients?
11. Which are exponents?
12. Which are terms?
13. Which are factors?
14. Which are additive or positive terms?
15. Which are subtractive or negative terms?

EXERCISE III.

1. Write down a added to b.

2. Write down a subtracted from b.

3. Write down the difference between a and b.

4. Express in three different ways the product of a and b.

5. Express in three different ways the division of a by b.

6. Express the *fourth power* of $a + b$.

7. Indicate in two different ways the extraction of the *fifth root* of a.

8. Indicate in two different ways the fourth power of the fifth root of ab.

9. Indicate that the sum of am and xy^2 is greater or less than the difference of a^3 and c.

10. Express the equality of the ratios a to m and xy to cf.

11. Write down the reciprocals of $\frac{a^2}{x}$, $\frac{1}{2}x^2$, $\frac{3}{4}\frac{x}{y^2}$, $a + b - c$, $(x + y)^{\frac{4}{7}}$.

12. What is the difference in meaning between $a + b^2$ and $a^2 + b^2$ and $(a + b)^2$?

13. What is the difference in meaning between ax^2y, axy^2, and a^2xy?

14. What is the difference in meaning between $mx^{\frac{2}{3}}$, $m^{\frac{2}{3}}x$, and $(mx)^{\frac{2}{3}}$.

15. What is the difference in meaning between $a - (x - y)$ and $(a - x) - y$?

16. What is the difference in meaning between $am - c$ and $am \sim c$?

17. Write down four homogeneous terms of 7 dimensions each.

18. Write down three homogeneous terms of 13 dimensions each.

19. Write any six *like* algebraic quantities.

20. Write down in an abbreviated form the product of a, a, a, a, m, m, m, $(x + y)$, $(x + y)$ and $am(x + y)$.

21. Resolve the expressions $7a^2$, $4a^3y^2$, a^3m^2y $(a + b)^2$, $a^4x^2(a - m)^3$ into their simple factors.

22. Express the division of the sum of mx^2 and y^3 by the square of the sum of a and b.

23. What is the coefficient and what the exponents of a and x in the expression ax?

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38. To find the numerical value of an algebraic expression, when the value of each letter entering into it is given :—

RULE.—*Substitute for each letter its numerical value, and perform upon the resulting numbers the operations indicated by the signs connecting them.*

Thus, in the following exercise, wherever a occurs in an expression, we write its assumed value, 1; for b we write 2; for c we write 3; for d we write 4; and for m we write 0: then we multiply, divide, add or subtract these quantities as directed by the connecting signs. For example, taking $a = 1$, $b = 2$, $c = 3$, and $m = 7$, we thus find the value of the expression :—

$$\sqrt{m(3a - 4c + 2b^3)} - \frac{bc + a}{m}$$

$$= \sqrt{7(3 \times 1 - 4 \times 3 + 2 \times 2^3} - \frac{2 \times 3 + 1}{7} = \sqrt{7(3 - 12 + 16)} - \frac{6 + 1}{7}$$

$$= \sqrt{7 \times 7} - \tfrac{7}{7} = \sqrt{49} - 1 = 7 - 1 = 6 \quad \textit{Ans.}$$

39. We are said to show that one algebraic quantity is numerically equal to another,

When by substituting the values for the individual letters in each we show that the numerical value of the first expression is the same as that of the other.

For example, if $a = 4$, $b = 3$, $d = 7$, and $f = 0$

$$a^2bdf + ab - d = 2d - (a + 2b) + 1$$

Here we at once throw out the quantity a^2bdf, because f being $= 0$, the whole quantity into which it enters as a factor must $= 0$, and, therefore, as an addend, it disappears; then substituting their values for the others,

$$4 \times 3 - 7 = 2 \times 7 - (4 + 2 \times 3) + 1$$
$$12 - 7 = 14 - (4 + 6) + 1$$
$$12 - 7 = 15 - 10$$
$$5 = 5$$

EXERCISE IV.

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $m = 0$, find the value of :—

1. $a^3 - 1$. 2. $c^3 \sim 3c$. 3. $ab + cd$.

4. $a^2b^2 \div (c - a)$. 5. $\sqrt{b + c + d}$. 6. $a^3m^2xd^3$

7. $6\,(a \sim c^2)$. 8. $(b^2d^2 - cm^2)^{\frac{2}{3}}$. 9. $(a + b)(d - m)^2$.

10. $4\{a - (d - c)\}^{\frac{2}{7}}$. 11. $(b^2c^2d^2)^{\frac{1}{2}}$. 12. $(d^2 - bc)^2\,(c^3 - bcd)^3$.

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13. Show that $\dfrac{a + 1}{\frac{1}{a} + 1} = a$, $\dfrac{b + 1}{\frac{1}{b} + 1} = b$, $\dfrac{c + 1}{\frac{1}{c} + 1} = c$, &c.

14. Show that $14a - (3b + c) < d^2 - b\,(b + c)$.

15. Show that $(a^2b - c^2d + abc)m = a^2b^2d^2m$.

16. Show that $\sqrt{ab^2c^3 - 4\,(b + d)\,c} > \{(b + c)\,(d^2 + c^2)\}^{\frac{1}{2}}$.

17. Show that $\dfrac{ab^2c^3 - bd}{a + b + c + d} = b\,(b + c) + ab^2c^3m$.

18. Show that $\dfrac{a^2c^2 + 2abcdm - (d - c)^2}{\sqrt[3]{2\,(d^2 + c^2) + b\,(c + d)}} = \{dc - (d + c + b + a)\}$

Find the numeral value of the following expressions :—

19. $(2 - b)\,(3a + 4b - c) + \{ab + (3d - 2c)\} - 4a\,(2c - 3b) - \{abc^2 - (3c + a)\} + \{abd - (c + d)\,a\}\,b$.

20. $(c^2 - a^2)\,(b^2 - m^2) + m\{bcd\,(a - b^2)d\} + 3a\{a + c(d - 3a)\}$.

21. $\{(a - b) + (c + d)\}^2 + \{(c + m) - (b - a)\}^3 - \{(m + d) + (2b - c)\}^2$.

22. $\sqrt{(a + c)\,d} + \sqrt[3]{c^2\,(a + b)} + \{2\,(d \div bc)^2 + (7d - b^2c)\}^{\frac{1}{3}} - (bcd + a)^{\frac{3}{2}}$.

23. $\dfrac{7\,(am)^{\frac{1}{2}} + 3\sqrt{d} - (bd + 4c)}{\frac{1}{3}\,abc + (cdm)^{\frac{2}{7}}} + \dfrac{a^2b^2c^2 - 7d + \{d^3\,(a + c)\}^{\frac{1}{2}}}{\{(b - a) + a^2\}\{d - (b + m)\}} - \sqrt[3]{abcd - d^2}$.

24. $\frac{1}{3}\{ab\,(a + b)\} - \frac{1}{4}\{bc\,(c + a)\} + \frac{1}{6}\{(ca - b)\,(a^2b + 3)\} + \frac{1}{7}\{(d + c)\,(1 + 3b - 2c + d)^2\}$

25. $\dfrac{c\,(a + b - c)^3 + 11\{(3a + 2c)\,(2a - b + \frac{1}{2}d)\}}{\{(3c + b) - \sqrt{d}\}\,(d + c + b^2 - m)} + \dfrac{\{(a + 3d)^2 - (c^3 + 5b) - (c + d)\}^{\frac{2}{3}}}{abm + \sqrt{dc^2} - a} + \dfrac{(2ab + cd - bd)\,(d + c)}{7(d + ab^2)}$

SECTION II.

ADDITION, SUBTRACTION, USE OF BRACKETS, MULTIPLICATION, AND DIVISION.

ADDITION.

40. When the quantities are similar and have the same sign:—

RULE.—*Add the coefficients, annex the literal part, and prefix the proper sign.*

(1)	(2)	(3)	(4)	(5)
$7a$	$-2cd$	$6(x+y)$	$-8(cd-a^2)$	$2a-3m+y-6\sqrt[3]{a+b}$
$3a$	$-3cd$	$2(x+y)$	$-4(cd-a^2)$	$3a-5m+6y-3\sqrt[3]{a+b}$
$5a$	$-cd$	$5(x+y)$	$-3(cd-a^2)$	$8a-7m+3y-5\sqrt[3]{a+b}$
$11a$	$-5cd$	$8(x+y)$	$-(cd-a^2)$	$5a-3m+2y-\sqrt[3]{a+b}$
$3a$	$-cd$	$(x+y)$	$-7(cd-a^2)$	$3a-2m+y$
$2a$	$-8cd$	$11(x+y)$	$-2(cd-a^2)$	$a-7m$
$31a$	$-20cd$	$33(x+y)$	$-25(cd-a^2)$	$22a-27m+13y-15\sqrt[3]{a+b}$

EXERCISE V.

Find the sum of :—

1. $3a$, $2a$, $9a$, $11a$, a and $17a$.
2. $-4ab^2$, $-7ab^2$, $-11ab^2$, $-ab^2$, and $-3ab^2$.
3. $3(a+b-c^2)$, $6(a+b-c^2)$, $2(a+b-c^2)$, $(a+b-c^2)$, and $7(a+b-c^2)$.
4. $4a(x-y^2)^{\frac{1}{3}}$, $9a(x-y^2)^{\frac{1}{3}}$, $3a(x-y^2)^{\frac{1}{3}}$, and $11a(x-y^2)^{\frac{1}{3}}$.
5. $3a-4y+7$, $6a-3y+3$, $5a-3y+3$, $7a-y+2$, and $6a-2y+8$.
6. $3(x+y)+7a-abc$, $5(x+y)+5a-3abc$, $2(x+y)+11a-7abc$, $(x+y)+2a-abc$, $2(x+y)+a-5abc$, and $3(x+y)+2a-3abc$.
7. $(a+b)x-(c+d)y-(d+f)z$, $5(a+b)x-6(c+d)y-7(d+f)z$, $2(a+b)x-3(c+d)y-4(d+f)z$, $4(a+b)x-5(c+d)y-6(d+f)z$, and $3(a+b)x-4(c+d)y-5(d+f)z$.
8. $a^2b^3x^{\frac{2}{3}}+a^3b^2x^{\frac{2}{3}}-a^2b^{\frac{2}{3}}x^3-a^3b^{\frac{2}{3}}x^2$, $3a^2b^3x^{\frac{2}{3}}+7a^3b^2x^{\frac{2}{3}}-5a^2b^{\frac{2}{3}}x^3-6a^3b^{\frac{2}{3}}x^2$, $7a^2b^3x^{\frac{2}{3}}+3a^3b^2x^{\frac{2}{3}}-5a^2b^{\frac{2}{3}}x^3-2a^3b^{\frac{2}{3}}x^2$, and $4a^2b^3x^{\frac{2}{3}}+a^3b^2x^{\frac{2}{3}}-2a^2b^{\frac{2}{3}}x^3-8a^3b^{\frac{2}{3}}x^2$.

41. When the quantities are similar, but all have not the same sign :—

RULE.—*Arrange the quantities so that similar terms shall be in the same vertical column. Add separately the positive and negative coefficients ; to the difference of these two sums prefix the sign of the greater and affix the common literal part.*

(1)	(2)	(3)	(4)
$4a$	$5a-3c$	$5ab+6cy-3$	$5(a+x)-3a^2xy+7\sqrt{a+b}$
$-7a$	$2a+4c$	$-8ab-3cy+11$	$9(a+x)-6a^2xy-8(a+b)^{\frac{1}{2}}$
$-3a$	$-3a+9c$	$-7ab+4cy-6$	$-7(a+x)+5a^2xy-6(a+b)^{\frac{1}{2}}$
$-2a$	$6a-5c$	$11ab-8cy+7$	$3(a+x)-3a^2xy-5(a+b)^{\frac{1}{2}}$
$5a$	$4a+3c$	$3ab-4cy+6$	$11(a+x)-5a^2xy+3(a+b)^{\frac{1}{2}}$
$6a$	$-7a-12c$	$-7ab+cy-1$	$-13(a+x)+6a^2xy-8\sqrt{a+b}$
$3a$	$7a-4c$	$-3ab-4cy+14$	$8(a+x)-6a^2xy-17(a+b)^{\frac{1}{2}}$

EXPLANATION.—In (1) the sum of the positive coefficients 6, 5, 4 = 15, sum of the negative coef. 2, 3, 7 = 12; then 15 ~ 12 = 3, which is positive, because 15, the greater, is the sum of the positive coefficient.

In (2), left hand column, the sum of pos. coef. 4, 6, 2, 5 = 17, and of neg. coef. 7, 3 = 10; then 10 ~ 17 = 7, which is pos. because 17 is pos. In right hand column sum of pos. coef. 3, 9, and 4 = 16, and of neg. coef. 12, 5, and 3 = 20; then 20 ~ 16 = 4, which is neg., because 20, the sum of the neg., is the greater.

EXERCISE VI.

Find the sum of :—

1. $a+b$ and $a-b$; $2a+b-c$ and $a-b+4c$; $4a-3b+c$ and $7b-8c$.

2. $2ab+3ay-cd$, $6ab-2ay+5cd$, $3ab-6ay+2cd$ and $-3ab-2ay+7cd$.

3. $5a^2x^{\frac{2}{3}}-3(a+b)-7x^{\frac{2}{3}}y+7$, $a^2x^{\frac{2}{3}}-7(a+b)-8x^{\frac{2}{3}}y-11$, and $-7a^2x^{\frac{2}{3}}+3(a+b)+3x^{\frac{2}{3}}y-16$.

4. $a+b-c-d$, $a-b-c+d$, $a-b+c-d$, $-a-b+c+d$, $-a+b-c+d$ and $a-b+c-d$.

5. $3xy+7ab-3$, $5xy+3ab+7$, $4xy-7ab+11$, and $-7xy+11ab+2$.

6. $3+7a-6b+c$, $7a+3-4b-2c$, $7b-3a-7+3c$, and $6c-2b+6-3a$.

7. $ab - xy + cd - m + c$, $6c - 3xy + 4m - cd - 3ab$, $5cd - 6m + 5c + 8ab - 3xy$, $5m + 6c - 3cd + 2xy - 3ab$ and $11xy - 3m - 2c + 3ab - 7cd$.

8. $5m^2x + 3xy - 7$, $7xy + 3 - 8m^2x + yz$, $17 - yz + 7xy - 11m^2x$ and $-11m^2x + 3xy + 4$.

9. $6m^{\frac{1}{3}}n^{\frac{2}{3}} - 9a^{\frac{2}{3}}d^{\frac{1}{2}} + 10m^{\frac{1}{2}}x^{\frac{3}{4}}$, $6a^{\frac{2}{3}}d^{\frac{1}{2}} - 6m^{\frac{1}{2}}x^{\frac{3}{4}} - m^{\frac{1}{3}}n^{\frac{2}{3}}$, $2a^{\frac{2}{3}}d^{\frac{1}{2}} - 3m^{\frac{1}{2}}x^{\frac{3}{4}} - 3m^{\frac{1}{3}}n^{\frac{2}{3}}$ and $-m^{\frac{1}{3}}n^{\frac{2}{3}} - m^{\frac{1}{2}}x^{\frac{3}{4}} + a^{\frac{2}{3}}d^{\frac{1}{2}}$.

10. $\sqrt{2} + \sqrt[3]{3} + \sqrt[4]{4} - \sqrt[5]{a} + c^{\frac{1}{6}}$, $11\sqrt{2} - 9\sqrt[3]{3} + 7\sqrt[5]{a} - 6\sqrt[4]{4} + \sqrt[6]{c}$, $-3\sqrt[3]{3} + 7\sqrt{2} + \sqrt[4]{4} - 7a^{\frac{1}{5}} + 8\sqrt[6]{c}$, $11a^{\frac{1}{5}} - \sqrt{2} + 3\sqrt[3]{3} + 7\sqrt[4]{4}$, and $9c^{\frac{1}{6}} - 4a^{\frac{1}{5}} + 11\sqrt[4]{4}$.

11. $3xy - 7ay + 2cx - x^{\frac{1}{2}} + 3\sqrt[3]{y}$, $2xy + 11\sqrt{x} - 7ay$, $13y^{\frac{1}{3}} - 11cx + 2ay$, $12\sqrt[3]{y} - 7x^{\frac{1}{2}} + 3cx - ay$, $11xy + 3ay + 6cx$ and $4xy - \sqrt{x} - 3\sqrt[3]{y}$.

12. $(ax + by - cz)^{\frac{1}{4}} - \sqrt{m+n} - (x-y)$, $7\sqrt{m+n} + 3(x-y) - \sqrt[4]{ax+by-cz}$, $7(x-y) + 8\sqrt[4]{ax+by-cz} - 11(m+n)^{\frac{1}{2}}$, $6\sqrt{m+n} + 17(ax+by-cz)^{\frac{1}{4}} - (x-y)$, $-12(ax+by-cz)^{\frac{1}{4}} - 3(x-y) + 4(m+n)^{\frac{1}{2}}$ and $7\sqrt{m+n} - 9\sqrt[4]{ax+by-cz} + 11(x-y)$.

42. When the quantities are unlike :—

RULE.—*Connect them together by their proper signs.*

(1)

$$\begin{array}{r} 3a \\ -4c \\ 7d \\ -5m \\ \hline \end{array}$$

$$\text{Sum} = 3a - 4c + 7d - 5m$$

(2)

$$\begin{array}{l} 5a + 3c - 6\sqrt{a+b} \\ 2m - 4a^2b + 3ab^2 \\ -6xy + 3a^{\frac{1}{2}}b^{\frac{1}{3}} \\ \hline \end{array}$$

$$\text{Sum} = 5a + 3c - 6\sqrt{a+b} + 2m - 4a^2b + 3ab^2 - 6xy + 3a^{\frac{1}{2}}b^{\frac{1}{3}}$$

43. When the quantities are partially similar :—

RULE.—*Add the similar quantities by Art.* 38, 39, *and to the partial sum, thus formed, affix the unlike quantities by their proper signs.*

(3)	(4)
$2a-4c+b$	$3ax^2y+7ay-10x^{\frac{1}{2}}+3a^2p$
$7a-3c+m$	$-5ax^2y \quad + 3x^{\frac{1}{2}}+7a^2p-mn$
$-9a+6c+3ab$	$-8ay \quad -13a^2p+qp$
$+7am$	$4ax^2y+ay+7x^{\frac{1}{2}} \quad +m^3$
$-c+b+m+3ab+7am$	$2ax^2y-3a^2p-mn+qp+m^3$

EXERCISE VII.

Find the sum of:—

1. $a+b$, $m+c$, $x+y$, and $3p$.

2. $2ap-3xy+4mn$, $5mn-3xz+7xy$, $3mn-5c^3+2ap$, and $-4ap-4xy-12mn$.

3. $3(a+b)+7(x-y)$, $7c+8(a+b)$, $11(x-y)+4x^2$, and $-16(x-y)-11(a+b)$.

4. $5x^2y-3y^2z+4$, $7y^2z-7m-3$, $5x^2y+3y^2z-a^2b$, and $6+7m-7y^2z$.

5. $a+b+c$, $3b-x+y$, $5(a+b)+3x$, $7c-3m^2n$, $5ab+6b-3y$, and $3(x+y)-8c$.

6. $7ax^2-3aby+7x^2y^2-3\sqrt{x}+5$, $7\sqrt{x}-3-7aby-6ax^2$, $3m-5\sqrt{a+y}+10aby$, $11-ax^2+5\sqrt{x}-9x^2y^2-7m$, and $2x^2y^2+4m-3\sqrt{x}+5$.

7. $x^3-3x^2y^2-y^3-zy+y^2$, $2y^3+7x^2y^2+3y^2-9$, $4yz+3+3x^3-5y^3+3x^2y^2$, $2y^3-6x^2y^2+2y$, $-3yz-x^2y^2+4y^2$, and $6-5y^2$.

8. $5(xy+xz-yz)^{\frac{1}{3}}+3(a+y)c-7a^2y$, $8(xy+xz-yz)^{\frac{1}{3}}-7(a+y)c+3m$, $8\sqrt[3]{xz+xy-yz}-4am$, $7(a+y)c-17\sqrt[3]{xz-yz+xy}$, $5am-3m-3(a+y)c-(xz-yz+xy)^{\frac{1}{3}}$ and x^2y-m^3.

SUBTRACTION.

44. THEOREM.—*The subtraction of any positive quantity is equivalent to the addition of the same quantity taken negatively; and the subtraction of any negative quantity is equivalent to the addition of the same quantity taken positively.*

DEMONSTRATION I. $a=a+b-b$ (Ax. VI); subtract $+b$ from each.
Then (Ax. III) $a-(+b)=a-b=a+(-b)$

" II. $a=a+b-b$ (Ax. VI); subtract $-b$ from each.
Then (Ax. III) $a-(-b)=a+b=a+(+b)$

45. To subtract one algebraic quantity from another.—

RULE.—*Change all the signs of the subtrahend or imagine them to be changed, and then proceed as in addition.*

NOTE.—Once the signs of the subtrahend are changed, the question is no longer one in subtraction, but is converted into an equivalent problem in addition.

From	$7a-13xy+27$	Equivalent question.	To	$7a-13xy+27$
Take	$5a-11xy+19$		Add	$-5a+11xy-19$
Remainder	$2a-\ \ 2xy+\ \ 8$		Sum	$2a-\ \ 2xy+\ \ 8$

(1)

From $9ab+\ \ 3xy-23$
Take $5ab-\ \ 7xy+17$

Rem. $4ab+10xy-40$

(2)

From $3x^2y-\ \ 7xy^2+3z^3-4$
Take $9x^2y+\ \ 4xy^2-5z^3+m$

Rem. $-6x^2y-11xy^2+8z^3-4-m$

(3)

From $2(x-y)+z^3(a-b)$
Take $-7(x-y)-a^2m+17$

Rem. $9(x-y)+z^3(a-b)+a^2m-17$

EXERCISE VIII.

1. From $4a^2y^2z-7xy^3+5az^2-\ \ 7xy+13m-11$
   Take $3a^2y^2z+4xy^3-6az^2-11xy-\ \ 7m-11$

2. From $3a-7c+4xy^2-7\sqrt{a-b^2}$
   Take $-11a+7c-\ \ m^2+6\sqrt{a-b^2}$

3. From $(a+b)\sqrt[3]{x^2-y}+7am^2-cd$
   Take $7am^2-3cd+4(a+b)(x^2-y)^{\frac{1}{3}}$

4. From $9(xy+y^2-z^3)^{\frac{1}{5}}+3\sqrt{x^2-y^2}+7a^{\frac{1}{2}}x^{\frac{1}{4}}-11\sqrt[3]{m}+17x\sqrt{a+b}$
   Take $5(xy-z^3+y^2)^{\frac{1}{5}}+17x(b+a)^{\frac{1}{2}}+3m^{\frac{1}{3}}-7a^{\frac{1}{2}}x^{\frac{1}{4}}+3(x^2-y^2)^{\frac{1}{2}}$

5. From $3+\sqrt{2}-5x+\sqrt[3]{4}-7y+8^{\frac{1}{4}}-6\sqrt{a-b}$
   Take $\sqrt{2}-13+4^{\frac{1}{3}}-6\sqrt[4]{8}-5x+16y+3(a-b)^{\frac{1}{2}}$

6. From $5a-6b-7c+4d-11e+7m-16x+y-7z$
   Take $4d-7z+5a-6b+m-5c+9x-11y+abcd$

USE OF BRACKETS.

46. Much difficulty is commonly experienced by a beginner in the management of brackets. His attention is therefore particularly directed to the following rules, remarks and exercise.

RULE 1.—*If any number of quantities, enclosed within brackets, be preceded by the sign* $+$, *the brackets may be struck out as of no value.*

This arises from the fact that when a quantity is added the signs of its terms are not changed.

RULE 2.—*If any number of quantities, inclosed within brackets, be preceded by the sign* $-$, *the brackets may be removed if all the included signs be first changed, i.e.* $+$ *into* $-$ *and* $-$ *into* $+$.

The necessity of thus changing the signs is manifest from the following illustration:—

$a-(b+c)$ means that we are to subtract the whole quantity $b+c$ from a. If we subtract b alone the remainder $a-b$ is too great by c, for we were to subtract the sum of b and c. Hence to obtain the correct remainder we must take c from $a-b$, but this gives $a-b-c$. Therefore $a-(b+c)=a-b-c$.

Again $a-(b-c)$ means that b is to be decreased by c, and the remainder taken from a. If now we take b from a, the remainder $a-b$ is too small by c, because we have subtracted a quantity too great by c. Hence to make the remainder $a-b$ what it ought to be we must add c, but this gives us $a-b+c$. Therefore $a-(b-c)=a-b+c$.

REMARK 1.—The learner must carefully note that in every case in which he meets with [or $\{$ or (he must look for the counter part) or $\}$ or] and that the above rules apply only to the signs of the quantities, simple or compound, included within the complete or outer bracket.

REMARK 2.—In removing the brackets from a quantity it is to be carefully remembered that the first sign within the bracket, when $+$, is always understood, and that the rules above given apply to it as well as to the other signs.

Ex. 1. Simplify $a+(b-c+d)$

OPERATION.

$$a+(b-c+d)=a+b-c+d$$

Ex. 2. Simplify $3a-(4c-d+3a-m)$

OPERATION.

$$3a-(4c-d+3a-m)=3a-4c+d-3a+m=-4c+d+m$$

Ex. 3. Simplify $3m-\{a+(c-m)\}$

OPERATION.

$$\begin{aligned}3m-\{a+(c-m)\}&=3m-a-(c-m)\\&=3m-a-c+m=4m-a-c\end{aligned}$$

Ex. 4. Simplify $1-\{1-(1-\{1-x\})\}$

OPERATION.

$$\begin{aligned}1-\{1-(1-\{1-x\})\}&=1-1+(1-\{1-x\})\\&=1-1+1-\{1-x\}\\&=1-1+1-1+x\\&=x\end{aligned}$$

Ex. 5. Simplify $(a-b)-\{-a-(b-a)\}-\{-(-\{-(-a+b)-c\}-b)-c\}$

OPERATION.*

$$\begin{aligned}&(a-b)-\{-a-(b-a)\}-\{-(-\{-(-a+b)-c\}-b)-c\}\\&=a-b+a+(b-a)+(-\{-(-a+b)-c\}-b)+c\\&=a-b+a+b-a-\{-(-a+b)-c\}-b+c\\&=a-b+a+b-a+(-a+b)+c-b+c\\&=a-b+a+b-a-a+b+c-b+c\\&=2c\end{aligned}$$

* Although, for the sake of illustrating each step, the process is here made to consist of several lines, the student is recommended to remove all the brackets at one operation, and thus to make only two distinct steps in the simplification.

EXERCISE IX.

Simplify the following expressions :—

1. $(a+m)-(c-6)+(5-m)-(a+e)+(c+3)-(5c+m)$
2. $(a-b-c)-(b-c-a)-(c-b-a)-(a+b+c)$
3. $(3a-4)-(6y-x)-(5a-4-6y)-(3a-4+\{-6\})$
4. $6-\{-(-\{-(-\{-(m)\})\})\}$

5. $(2a-3c+4d)-\{5d-(m+3a)\}+\{5a-(-4-d)\}-\{3a-(4a-5d-4)\}$

6. $m^2-(c^2-a^2)-\{-m^2-(-2a^2)\}-\{-(-5m^2-\{-(a^2-c^2+3m^2)-c^2\}-m^2)-2a^2\}$

7. $1-(-1)-\{-(-1)\}-\{-(-\{-(-1)-1\})-1\}$

8. $a^2+2x-\{a^2-(2x^2-\{-m^2-(a^2+2x-\{-m^2-(3a^2+3x+3m^2)\})\}-2m^2)-a^2\}$

9. $(a^2bc+3c^2)+3a^2bc-(m+c)-\{-(4a^2bc+c)-(-3c^2-m)\}$

10. $3a-(2a+1)+\{a-(2-a)\}-\{-1-(-a-\{-2-a+(-1)\}-2a)\}$

11. $(-a-b-c)+(a-c)-(c-a)-\{-(+\{+(+\{+(+\{-a\}-b-c)-a\}-3b)\}-3b-2c)-2a\}$

12. $\{(am+c)-7\}+\{(5-7am+c)\}-\{-3a-(-4am-\{-c-(-9-3c-4a)\}-6)-5am\}$

---

47. It is frequently found necessary in the performance of algebraic operations to inclose two or more simple terms within brackets so as to deal with them as constituting one quantity. In placing any given terms within a bracket, attention must be paid to the following rules:—

RULE I.—*Any term whatever may be selected as the first term within the bracket, remembering that the sign of that term must be placed* before *the bracket.*

RULE II.—*If the sign thus placed before the bracket be* +, *the other terms may be at once placed within the bracket, each preceded by its proper sign; but if the sign thus placed before the bracket be* −, *then in placing the other terms within the bracket we must change the sign of each, i.e.,* + *into* − *and* − *into* +.

NOTE.—The signs are thus changed when the terms are put into a bracket preceded by the sign −, in view of the fact that when the brackets are struck out this − sign has the effect of changing the included signs back again to their original form.

Ex. 1. Inclose $a-b-c+d$ in a pair of brackets.

OPERATION.

$$+a-b-c+d=+(a-b-c+d)$$
$$\text{or } =-(b-a+c-d)$$
$$\text{or } =-(c-a+b-d)$$
$$\text{or } =+(d+a-b-c)$$

Ex. 2.—Inclose $a-b+c-d-m+f$, in alphabetical order, in brackets, using an outer bracket inclosing two pair of inner brackets.

OPERATION.

$$
\begin{aligned}
a-b+c-d-m+f &= \{(a-b+c)-(d+m-f)\} \\
\text{or } &= \{(a-b)+(c-d-m+f)\} \\
\text{or } &= \{(a)-(b-c+d+m-f)\} \\
\text{or } &= \{(a-b+c-d)-(m-f)\} \\
\text{or } &= \{(a-b+c-d-m)+(f)\}
\end{aligned}
$$

EXERCISE X.

Express $a-b+c-d-e+m-f-r-s+v+w+x$ in brackets.

1. Taking the terms *two* together.
2. Taking the terms *three* together.
3. Taking the terms *four* together.
4. Taking the terms *six* together.
5. Three together, using an inner bracket after the model, $\{* \pm (* \pm *)\}$
6. Three together, using an inner bracket after the model, $\{(* \pm *) \pm *\}$
7. Four together, using an inner bracket after the model, $\{* \pm (* \pm * \pm *)\}$
8. Four together, using an inner bracket after the model, $\{(* \pm * \pm *) \pm *\}$
9. Four together, using an inner bracket after the model, $\{* \pm (* \pm *) \pm *\}$
10. Six together, using an inner bracket after the model, $\{* \pm * \pm * \pm (* \pm * \pm *)\}$
11. Six together, using an inner bracket after the model, $\{(\pm * \pm * \pm * \pm *) \pm * \pm *\}$
12. Six together, using two inner brackets after the model, $\{* \pm (* \pm *) \pm * \pm (* \pm *)\}$

NOTE.—The asterisk is used merely to denote the position to be occupied by the given letters with reference to the brackets, the sign $\pm$, read *plus* or *minus*, implies here that the student is to determine which *one* of these signs is to be employed.

48. A number or a letter written directly before or after a bracket, inclosing one or more quantities, implies that each of the included terms is to be multiplied by that number or letter. So the line that separates the numerator and denominator of an algebraic fraction acts as a vinculum in uniting the terms of the numerator into one quantity, and hence when the several terms of the numerator are written separately the denominator must be placed under each.

Ex. 1. Remove the bracket from $6\,(a - am + by^2 - c)$.

OPERATION.

$$6\,(a - am + by^2 - c) = 6a - 6am + 6by^2 - 6c$$

Ex. 2. Remove the bracket from $4\,\{a - b - (cx + dy - b^3)\,a\}\,m$

OPERATION.

$$\begin{aligned} 4\,\{a - b - (cx + dy - b^3)\,a\}\,m &= 4m\,\{a - b - (cx + dy - b^3)\,a\} \\ &= 4am - 4bm - 4m\,(cx + dy - b^3)\,a \\ &= 4am - 4bm - 4am\,(cx + dy - b^3) \\ &= 4am - 4bm - 4acmx - 4admy + 4ab^3m \end{aligned}$$

Ex. 3. Remove the vinculum from $\dfrac{3a - m - (c^2 - m^2 + x)y}{2b^3\sqrt{c}}$

$$\begin{aligned} \frac{3a - m - (c^2 - m^2 + x)y}{2b^3\sqrt{c}} &= \frac{3a}{2b^3\sqrt{c}} - \frac{m}{2b^3\sqrt{c}} - \frac{c^2y - m^2y + xy}{2b^3\sqrt{c}} \\ &= \frac{3a}{2b^3\sqrt{c}} - \frac{m}{2b^3\sqrt{c}} - \frac{c^2y}{2b^3\sqrt{c}} + \frac{m^2y}{2b^3\sqrt{c}} - \frac{xy}{2b^3\sqrt{c}} \end{aligned}$$

NOTE.—In the first step of this operation, when the bracket inclosing the last three terms is struck out, the included signs are not changed, because the vinculum written under these terms still binds them into one, but when in the next step this vinculum is removed, the *minus* sign preceding it has the effect of changing the signs of the terms as exhibited in the operation.

EXERCISE XI.*

Remove the brackets and vincula from the following expressions :—

1. $3(a - b)$; $4x(a + b^2 - x^3)$; $3p^2x\,(1 - b - c^2)$
2. $m(a - b^2 + mp) + x^2\,(1 - 3a - b) - m^2x^2\,(3 - b - m^2x)$

---

* See Arts. 52, 53, and 57.

3. $3\{1-(x-y)a\}+4\{1+(a-b+y)x\}-c^2\{a-(-3-m)y\}$

4. $a\{a(m-n)-c(p-q)\}+c\{c(-m+n)+a(-p+q)\}$

5. $a-b-\dfrac{x+y-(c-d-m)}{z^3}$

6. $m+\dfrac{a-(b-c-d)}{xyz}$

7. $a\{(m-y)x-c(a+b)\}+ay-\dfrac{6a-(m-3p)}{2a-c}$

8. $3b\{-(a-c)d+(m-n)f\}-\dfrac{1-\{2(1-c)+3(1-m)-4(1-p)\}}{5x^2}$

---

49. Two or more terms of an algebraic expression that have a common factor are often written in an abbreviated form by the aid of brackets, placing the factor common to the several terms directly before or after the bracket, and the remaining part of each term with its proper sign within.

Ex. 1.—Collect the coefficients of x^2yz in the following expression into one quantity: $5ax^2yz-3x^3yz+5a^2m^2x^2yz+3abc^2x^2yz-x^2yz$.

OPERATION.

$$5ax^2yz-3x^3yz+5a^2m^2x^2yz+3abc^2x^2yz-x^2yz$$
$$=(5a-3x+5a^2m^2+3abc^2-1)x^2yz$$

50. Any factor of an algebraic term may be regarded as the coefficient of the remaining factor. This is at once evident from the meaning of the expression *coëfficient* = *con* "together with," and *efficiens* "making" or "operating," i. e., the part which *coöperates* with the remainder to make the complete term.

Thus, in the term $3abxy$, 3 is the coef. of $abxy$; $3a$ is the coef. of bxy; $3ab$ is the coef. of xy; $3abx$ is the coef. of y; $3aby$ is the coef. of x; $abxy$ is the coef. of 3; $3xy$ is the coef. of ab, &c., &c.

51. When terms involving brackets are to be added or subtracted it is commonly best first to strike out the brackets by Art. 46, and then after performing the addition or subtraction re-bracket the terms, if necessary.

Ex. 1. Add $2a(x-y+3)$, $5(m-c^2-ax)$, and $2(a+ay-4m)$

OPERATION.

$$\begin{aligned} 2a(x-y+3) &= 2ax - 2ay + 6a \\ 5(m-c^2-ax) &= -5ax \qquad\qquad + 5m - 5c^2 \\ 2(a+ay-4m) &= \underline{\qquad\quad 2ay + 2a - 8m \qquad} \\ \text{Sum} &= -3ax \qquad + 8a - 3m - 5c^2 \\ &= -3(ax+m) + 8a - 5c^2 \end{aligned}$$

Ex. 2. From $p(x-y)+q(y-z)$ take $a(x-z)-b(y+z)$

OPERATION.

$$\begin{aligned} p(x-y)+q(y-z) &= px - py + qy - qz \\ a(x-z)-b(y+z) &= \underline{ax - az - by - bz \qquad\qquad} \\ \text{Diff.} &= px - py + qy - qz - ax + az + by + bz \\ &= px - ax - py + qy + by - qz + az + bz \\ &= (p-a)x - (p-q-b)y - (q-a-b)z \end{aligned}$$

EXERCISE XII.*

Find the value of:—

1. $3(am-x+y)+5a(x+3y)+2(a-y)m+4x(a+1)$.
2. $(a-x+y)m+3(m+a)x+4(a-y)+3(a+x)y$.
3. $7(a+b-c)-5(b+x-bc)-3(m-a-c)$.
4. $(a+m)x-3(am+c)xy+2(a-cm)y^2$ added to $(x+y^2)a+(c+a)xy-(b+f)y^2$.
5. $3(x+y+z)am+2c(x+z)+(y-z)ac$ subtracted from $3(a-b+c)y-(2m-c)x-3m(ax+ay-az)$.
6. $2a(p+xy)c-3(m-2xy+y^2)c-3a(y+c)$ subtracted from $11(a+b)my-3xy(a-b+c)$.

MULTIPLICATION.

52. THEOREM.—*Quantities having like signs, give, when multiplied together, a product which is positive; and quantities having unlike signs, give, when multiplied together, a product which is negative.*

Or, as it is sometimes expressed for the sake of brevity,—

In Multiplication, like signs give PLUS, *and unlike signs,* MINUS.

* See Arts. 52 and 53.

DEMONSTRATION I. $+a \times +b$ means that $+a$ is to be taken in an additive sense, i. e., is to be added as often as there are units in b. But $+a$ added once gives $+a$; $+a$ added two times gives $+2a$; $+a$ added three times gives $+3a$, and so on. Hence $+a$ added b times gives $+ab$, that is, $+a \times +b = +ab$.

II. $-a \times +b$ means that $-a$ is to be taken in an additive sense as often as there are units in b, but $-a$ added once gives $-a$; $-a$ added two times gives $-2a$; $-a$ added three times gives $-3a$, and so on. Hence $-a$ added b times gives $-ab$ that is $-a \times +b = -ab$.

Otherwise, $-a+a=0$; multiply each of these equals by $+b$.

Then $-a \times +b + ab = 0$; subtract $+ab$ from each of these equals.

Then $-a \times +b = -ab$, which was to be proved.

III. $+a \times -b$ is equivalent to $-b \times +a$ since quantities connected by the sign of multiplication can be read in any order whatever.

But $-b \times +a = -ab$ by last case. Therefore also $+a \times -b = -ab$.

IV. $-a+a=0$; multiply each of these equals by $-b$.

Then $-a \times -b - ab = 0$; add $+ab$ to each of these equals.

Then $-a \times -b = +ab$, which was to be proved.

53. THEOREM II.—*Different powers of the same quantity are multiplied together by adding their exponents.*

DEMONSTRATION.—$a^4 \times a^3 = aaaa \times aaa = aaaaaaa = a^7 = a^{4+3}$, and the same is true in all other cases, hence generally $a^m \times a^n = a^{m+n}$.

CASE I.

54. When multiplicand and multiplier are both simple algebraic quantities,

RULE.—*Multiply together the numerical coefficients and write the letters in juxtaposition after this product.*

Thus $3ab \times 5cy = 3 \times 5 \times abcy = 15abcy$; $-2ab \times 3c = -6abc$; $2xy \times -11m = -22mxy$; $-4xy \times -7am = 28amxy$.

CASE II.

55. When the multiplier is a simple quantity and the multiplicand is a polynomial,

RULE.—*Multiply each term of the multiplicand by the multiplier, and connect the several partial products by their proper signs.*

Ex. 1.

$$\begin{array}{ll} \text{Multiplicand,} & 4ax - 2ay + 3x^2y^2 \\ \text{Multiplier,} & 2axy \\ \hline \text{Product,} & 8a^2x^2y - 4a^2xy^2 + 6ax^3y^3 \end{array}$$

Ex. 2.

$$\begin{array}{ll} \text{Multiplicand,} & 4am^2 - 3acx - 4xy + 7 \\ \text{Multiplier,} & -3ay^2 \\ \hline \text{Product,} & -12a^2m^2y^2 + 9a^2cxy^2 + 12axy^3 - 21ay^2 \end{array}$$

CASE III.

56. When both multiplier and multiplicand are polynomials,

RULE.—*Multiply each term of the multiplicand by each term of the multiplier, and add the several partial products together.*

Ex. 3.

$$\begin{array}{l} a^2 - ab - b^2 \\ a - b \\ \hline a^3 - a^2b - ab^2 \\ \quad - a^2b + ab^2 + b^3 \\ \hline a^3 - 2a^2b \qquad + b^3 \end{array}$$

Ex. 4.

$$\begin{array}{l} 3ax^2 - 3a^2x + 2a^2x^2 \\ 5a \quad - 2x \\ \hline 15a^2x^2 - 15a^3x + 10a^3x^2 \\ \quad - 6ax^3 + 6a^2x^2 - 4a^2x^3 \\ \hline 21a^2x^2 - 4a^2x^3 - 6ax^3 - 15a^3x + 10a^3x^2 \end{array}$$

Ex. 5.

$$\begin{array}{l} 2ab^2 - a^2b^2 + a^3b^3 \\ 3ab - 2ab^2 - 3a^2b \\ \hline 6a^2b^3 - 3a^3b^3 + 3a^4b^4 \\ \quad - 4a^2b^4 + 2a^3b^4 - 2a^4b^5 \\ \qquad - 6a^3b^3 + 3a^4b^3 - 3a^5b^4 \\ \hline 6a^2b^3 - 9a^3b^3 - 4a^2b^4 + 3a^4b^4 + 3a^4b^3 - 2a^4b^5 - 3a^5b^4 \end{array}$$

Ex. 6.

$$\begin{array}{l} a^2 - 2ab + b^2 \\ a^2 + 2ab + b^2 \\ \hline a^4 - 2a^3b + a^2b^2 \\ \quad 2a^3b - 4a^2b^2 + 2ab^3 \\ \qquad a^2b^2 - 2ab^3 + b^4 \\ \hline a^4 \qquad - 2a^2b^2 \qquad + b^4 \end{array}$$

Ex. 7.
$$\begin{array}{l}
x^2 - (a-b)x + ab \\
x \;\; - m \\
\hline
x^3 - (a-b)x^2 + abx \\
\quad\; - mx^2 \qquad + (ma - mb)x - abm \\
\hline
x^3 - (a-b+m)x^2 + (ma - mb + ab)x - abm
\end{array}$$

Ex. 8.
$$\begin{array}{l}
x^3 - ax^2 - bx + c \\
x \;\; - m \\
\hline
x^4 - ax^3 - bx^2 + cx \\
\quad\; - mx^3 + amx^2 + bmx - cm \\
\hline
x^4 - (a+m)x^3 - (b - am)x^2 + (c + bm)x - cm
\end{array}$$

Exercise XIII.

1. Multiply $a^2 - 2ay + y^2$ by $a^2 - 2ay + 2y^2$; and $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 + 2ab + b^2$.

2. Multiply $2a^2m^2 + 12amxy + 9x^2y^2$ by $am - xy$; and $3a^2x - 3ax^2$ by $3a^2x^3 - x^2 - 1$.

3. Multiply $a^4 - a^3m + a^2m^2 - am^3 + m^4$ by $a + m$; and $2a^2 - 2axy + 2y^2$ by $a^2 - ax + y^2$.

4. Multiply $x^2 - 3x - 7$ by $x - 4$ and $a^2 + a^4 + a^6$ by $a^2 - 1$.

5. Multiply $a^3 + 2a^2b + 3ab^2 + 4b^3$ by $a^2 - 2ab - 3b^2$.

6. Multiply $ab - ac + bc$ by $ab + ac - bc$.

7. Multiply $a^4 - 2a^3b - 3a^2b^2 - 2ab^3 + b^4$ by $a^2 + 2ab + b^2$.

8. Multiply $3x^2 - 2abx - 2a^2b^2$ by $x + 2ab$; and $x^2 + 2x - 3$ by $x^2 - x + 1$.

9. Multiply $x^4 + 2x^3 + 3x^2 + 2x + 1$ by $x^4 - 2x^3 + 3x^2 - 2x + 1$

10. Multiply $3y^3 + 2x^2y^2 + 3x^2$ by $2y^3 - 3x^2y^2 + 5x^3$; and $a^m + b^m$ by $a^n + b^n$.

11. Multiply $2a + 3$, $3a + 4$, $5a^2 - 2$, and $a - 3$ together.

12. Multiply $ax + by$ by $ax + cy$; and $a^m - b^n + c^p$ by $a^{m+1} - b^{n-p}$.

13. Multiply $a^m - c^p + q^r$ by $a^2 - m^3 + x^a$.

14. Multiply $a^2 - ax + x^2$ by $a^3 - a^2x + ax^2 - x^3$.

15. Multiply $2a - b$, $3b + c$, $2c - m$, and $3m - x$ together.

DIVISION.

57. Division is the process of resolving a given quantity into two factors when one of the latter is given. As in Arithmetic, the given quantity to be resolved or divided is called the *dividend*, the given factor is called the *divisor*, and the factor to be obtained, the *quotient*.

Since the divisor × quotient = dividend, the sign of the quotient must be such that the sign of its product by the divisor shall be the sign of the dividend.

Thus, $\frac{+ab}{+b} = +a \because +a \times +b = +ab$; $\frac{+ab}{-b} = -a \because -a \times -b = +ab$;

$\frac{-ab}{-b} = +a \because -b \times +a = -ab$; $\frac{-ab}{+b} = -a \because -a \times +b = -ab$.

Hence, the rule of signs for division is the same as for multiplication; that is, like signs in divisor and dividend give PLUS *in the quotient, unlike signs in divisor and dividend give* MINUS *in the quotient.*

58. Since $a^4 \times a^3 = a^{4+3} = a^7$, it follows that $a^7 \div a^4 = a^3$, that is, $a^7 \div a^4 = a^{7-4} = a^3$; or generally, since $a^m \times a^n = a^{m+n}$, it follows that $a^{m+n} \div a^m = a^n$ or $a^{m+n} \div a^n = a^m$.

Hence, one power of any quantity is divided by another power of the same quantity, by subtracting the exponent of the divisor from the exponent of the dividend.

Thus, $a^6b^5 \div a^2b^2 = a^4b^3$; $x^3z^5 \div xz^5 = x^2$; $ab^2c^3m^4 \div bm^3 = abc^3m$, &c.

CASE I.

59. When both dividend and divisor are simple quantities or monomials,

RULE.—*Divide separately the coefficient of the dividend by the coef. of the divisor, and the literal part of the dividend by the literal part of the divisor; write the partial quotients thus obtained in juxtaposition, and prefix the proper sign.*

Thus, $14a^7b^2c^8 \div -7a^3bc^4$, $14 \div 7 = 2$, and $a^7b^2c^8 \div a^3bc^4 = a^4bc^4$, and the quotient is $-2a^4bc^4$, because the signs of divisor and dividend are unlike.

Similarly $-21a^2bx \div 3a^2b = -7x$; $-18xy^2z^3 \div -2xz^2 = 9y^2z$, &c.

NOTE.—If both coef. and literal part of the divisor are not contained as factors in the dividend, we can only indicate the division by writing the two quantities in the form of a fraction.

For example, $7ab^2cx^3 \div 11my$ can only be expressed thus, $\dfrac{7ab^2cx^3}{11my}$

But when we have thus expressed the quotient we can cancel any factors that are common to both numerator and denominator.

Thus, $24a^2xy^2 \div 15axz^2 = \dfrac{24a^2xy^2}{15axz^2} = \dfrac{3ax \times 8ay^2}{3ax \times 5z^2} = \dfrac{8ay^2}{5z^2}$

EXERCISE XIV.

Find the quotients of:

1. $15abc^2 \div 5ac$; $42ax^3y^5 \div 7axy^4$; $24a^2xy \div 8axy$; $-20x^2y^4z^{10} \div 20xy^3z^7$.
2. $-14ab^2cm^4 \div 7abm^3$; $-14abx^3 \div 14bx$; $-27mx^3y \div -3x^2$; $-12x^7y \div -4x^3y$.
3. $12ab^2c \div 20axy$; $-17abx^2 \div 11amx$; $-21abx^3y \div -35bx^2z^4$; $ab^3cf \div -16acfx^2$.

CASE II.

60. When the divisor is a simple quantity but the dividend a compound quantity, i. e., a polynomial,

RULE.—*Divide each term of the polynomial by the divisor, as directed in* Case I, *and connect the several partial quotients thus obtained by their proper signs.*

EXAMPLE.—Divide $4a^2b^2c - 3abc^2 + 12ab^3cx - 8aby^2$ by $-4ab$.

Here $\dfrac{4a^2b^2c - 3abc^2 + 12ab^3cx - 8aby^2}{-4ab} = \dfrac{+4a^2b^2c}{-4ab}$, and $\dfrac{-3abc^2}{-4ab}$, and $\dfrac{+12ab^3cx}{-4ab}$, and $\dfrac{-8aby^2}{-4ab} = -abc$, and $+\dfrac{3c^2}{4}$, and $-3b^2cx$, and $+2y^2 = -abc + \dfrac{3c^2}{4} - 3b^2cx + 2y^2$.

EXERCISE XV.

Find the quotients of :—

1. $12axy^2 - 27abc^2 + 12ax^2y - 8acm \div 4acx$.
2. $21xy^2 - 11a + 14x^2y - 49y^2 \div 35axy$.
3. $-64a^4m - 16a^2m^2 + 24a^3m - 40m^2xy \div -16a^2m$.
4. $3abc + 4a^2c^2 - 16axy^2 - 30a^2m \div -12mxy$.

CASE III.

61. When both divisor and dividend are polynomials,

RULE I.—*Arrange the terms of both divisor and dividend, so that the different powers of some one letter (which is common to both of them) may succeed each other in the order of their indices, and place the divisor thus arranged to the left of the arranged dividend, as in arithmetical division.*

II.—*Divide by Case I. the* FIRST TERM *of the dividend by the* FIRST TERM *of the divisor, and place the result with its proper sign in the quotient,*

III.—*Multiply the whole divisor by the term placed in the quotient, set the product beneath the dividend, and subtract.*

IV.—*To the remainder bring down as many terms from the dividend as the case may require; again divide the first term of this partial dividend by the first term of the divisor, and place the result with its proper sign as second term of the quotient; multiply and subtract as before, and proceed thus till all the terms are brought down.*

EXAMPLE 1. $a + b \,)\, a^2 + 2ab + b^2 \,(\, a + b$

$$\begin{array}{r} a^2 + ab \\ \hline ab + b^2 \\ ab + b^2 \\ \hline \end{array}$$

EXPLANATION.—The terms are already properly arranged in both divisor and dividend, since the powers of a follow one another in regular descending order. Then a^2 (first term of dividend) $\div a$ (first term of divisor) gives $+a$ as result, and we place this in the quotient. Next $(a+b) \times a = a^2 + ab$ which we subtract from the dividend, and to the remainder $+ab$ we bring down b^2, the other term of the dividend. Next $+ab$ (first term of partial dividend) $\div a$ (first term of divisor) gives b for second term of quotient. Lastly $(a+b) \times b = ab + b^2$ which we subtract and find that there is no remainder.

Ex. 2. $3ab+4b^2)-ab^3+6a^2b^2-12b^4$

$$
\begin{array}{rl}
3ab+4b^2) & 6a^2b^2-ab^3-12b^4(2ab-3b^2 \\
 & \underline{6a^2b^2+8ab^3} \\
 & \qquad -9ab^3-12b^4 \\
 & \qquad \underline{-9ab^3-12b^4}
\end{array}
$$

EXPLANATION.—Here we see that the terms as given are not properly arranged, since in the divisor the exponents of a are arranged in descending order, while in the dividend they are not; moreover the exponents of b in the divisor follow one another in ascending order, but in the dividend they follow one another irregularly. We first then arrange them properly, and then proceed to divide as follows: $6a^2b^2 \div 3ab = +2ab$, which we place in the quotient, $(3ab+4b^2)\times 2ab = 6a^2b^2+8ab^3$, which subtracted from the dividend gives a remainder $-9ab^3-12b^4$. Next $-9ab^3 \div 3ab = -3b^2$; $(3ab+4b^2)\times -3b^2 = -9ab^3-12b^4$, which subtracted leaves no remainder.

Ex. 3.
$$
\begin{array}{rl}
3a-6\ ) & 6a^4-96\ (\ 2a^3+4a^2+8a+16 \\
 & \underline{6a^4-12a^3} \\
 & \quad 12a^3-96 \\
 & \quad \underline{12a^3-24a^2} \\
 & \qquad 24a^2-96 \\
 & \qquad \underline{24a^2-48a} \\
 & \qquad\quad 48a-96 \\
 & \qquad\quad \underline{48a-96}
\end{array}
$$

Ex. 4. $x^2-xy+y^2)\ x^2y^2+x^4+y^4$

$$
\begin{array}{rl}
x^2-xy+y^2\ ) & x^4+x^2y^2+y^4\ (\ x^2+xy+y^2 \\
 & \underline{x^4-x^3y+x^2y^2} \\
 & \quad x^3y+y^4 \\
 & \quad \underline{x^3y-x^2y^2+xy^3} \\
 & \qquad x^2y^2-xy^3+y^4 \\
 & \qquad \underline{x^2y^2-xy^3+y^4}
\end{array}
$$

Ex. 5.

$$
\begin{array}{l}
a^2-2ax+x^2)\ a^4-4a^3x+6a^2x^2-4ax^3+4x^4\ (a^2-2ax+x^2+\dfrac{3x^4}{a^2-2ax+x^2} \\
\qquad a^4-2a^3x+a^2x^2 \\
\hline
\qquad -2a^3x+5a^2x^2-4ax^3 \\
\qquad -2a^3x+4a^2x^2-2ax^3 \\
\hline
\qquad\qquad a^2x^2-2ax^3+4x^4 \\
\qquad\qquad a^2x^2-2ax^3+x^4 \\
\hline
\qquad\qquad\qquad 3x^4 = \text{rem.}
\end{array}
$$

Ex. 6.

$$
\begin{array}{l}
1+a)\ a^2+2a+1\ (a^2-a^3+a^4-a^5+\dfrac{a^6+2a+}{1+a} \\
\qquad a^2+a^3 \\
\hline
\qquad -a^3+2a \\
\qquad -a^3-a^4 \\
\hline
\qquad\qquad a^4+2a \\
\qquad\qquad a^4+a^5 \\
\hline
\qquad\qquad -a^5+2a \\
\qquad\qquad -a^5-a^6 \\
\hline
\qquad \text{Rem.} = a^6+2a+1
\end{array}
$$

NOTE.—In Examples 5 and 6 the division does not terminate, or in other words, the dividend is not exactly divisible by the divisor, and we write the remainder as the numerator of a fraction having the divisor for denominator. In Example 6, however, this inconvenience arises from the fact that the terms of both divisor and dividend are not arranged according to rule, for if we had arranged the dividend thus $(1+2a+a^2)$ we should have obtained $1+a$ for the quotient. The student then must be careful to remember that the divisor and dividend must be arranged either both according to the ascending or both according to the descending powers of the principal letter, or *letter of reference*, as it is called; and that not only at starting, but throughout the whole process he must take care to arrange the partial dividends according to the same plan as that adopted in the divisor.

EXERCISE XVI.

Find the quotients of :—

1. $x^2-2xy+y^2$ divided by $x-y$; and $a^3+3a^2b+3ab^2+b^3$ divided by $a+b$.
2. $m^4+4m^3x+6m^2x^2+4mx^3+x^4$ divided by $m^2+2mx+x^2$.
3. $9x^6-46x^5+95x^2+150x$ divided by x^2-4x-5.

4. $a^3 + 5a^2b + b^3 + 5ab^2$ divided by $a + b$; and $-1 + x^3y^3$ divided by $-1 + xy$.

5. $x^6 + 10x - 33$ divided by $3 + x^2 - 2x$.

6. $a^8 + 2a^6m^3 - 2a^4m^4 - 2a^7m + m^8 - 2am^7 + 2a^3m^6$ divided by $a^3 + m^3 - a^2m - am^2$.

7. 1 divided by $1 + a$; a divided by $1 - a$; $1 - m$ divided by $m + 1$; and $1 - 2x + 3x^2 \div 1 + x - x^2$.

8. $6a^4 - 10a^3m - 22a^2m^2 + 46am^3 - 20m^4$ divided by $4am + 3a^2 - 5m^2$.

9. $4a^5 - 16a^3b^2 + 10a^2b^3 + 15ab^4 - 25b^5$ divided by $2a^2 - 5b^2$.

10. $a^3 + b^3 + c^3 - 3abc$ divided by $a^2 + b^2 + c^2 - bc - ac - ab$.

11. $144x^4 - 145x^2y^2 + 36y^4$ divided by $4x + 3y$.

12. $2a^{2m} + 2a^mb^p - 4a^mc^n - 3a^mb - 3b^{p+1} + 6bc^n$ divided by $a^m + b^p - 2c^n$.

NOTE.—If the teacher is desirous of giving his pupils a greater number of questions in division he can find material for such in Exercise XIII, in which the product may be regarded as the dividend, and either the multiplier or multiplicand as the divisor. Similarly, the questions in Exercise XVI. may be made to furnish additional material for practice in multiplication.

DIVISION BY DETACHED COEFFICIENTS.

62. It is sometimes convenient in division, as also in multiplication, to employ only the coefficients. The mode of proceeding is shown in the following rule and illustration :—

RULE.—*Having arranged the divisor and dividend as in ordinary division, omit the letters, and set down the coefficients, each preceded by its proper sign, and place zero for every term of either divisor or dividend that may chance to be absent.*

Proceed with these coefficients as in ordinary division, and the result will be the coefficients of the quotient with their proper signs; the literal part to attach to each of these is easily determined by inspection.

Ex. 1. Divide $9x^4 - 144$ by $3x - 6$.

OPERATION.

$$
\begin{array}{l}
3 - 6 \;)\; 9 + 0 + 0 + 0 - 144 \;(\; 3 + 6 + 12 + 24 \\
\qquad\quad \underline{9 - 18} \\
\qquad\qquad\quad 18 + 0 \\
\qquad\qquad\quad \underline{18 - 36} \\
\qquad\qquad\qquad\quad 36 + 0 \\
\qquad\qquad\qquad\quad \underline{36 - 72} \\
\qquad\qquad\qquad\qquad\quad 72 - 144 \\
\qquad\qquad\qquad\qquad\quad \underline{72 - 144}
\end{array}
$$

Hence the quotient $= 3x^3 + 6x^2 + 12x + 24$.

EXPLANATION.—We place three ciphers in the dividend to occupy the places of the absent terms x^3, x^2, and x. We ascertain the literal parts to attach, by observing that $x^4 \div x = x^3$, which we place after the first coefficient, and the others of course follow in regular order.

Ex. 2. Divide $x^6 + 4x^5 - 8x^4 - 25x^3 + 35x^2 + 21x - 28$ by $x^2 + 5x + 4$.

OPERATION.

$$
\begin{array}{l}
1 + 5 + 4 \;)\; 1 + 4 - 8 - 25 + 35 + 21 - 28 \;(\; 1 - 1 - 7 + 14 - 7 \\
\qquad\qquad\quad \underline{1 + 5 + 4} \\
\qquad\qquad\qquad - 1 - 12 - 25 \\
\qquad\qquad\qquad \underline{- 1 - 5 - 4} \\
\qquad\qquad\qquad\quad - 7 - 21 + 35 \\
\qquad\qquad\qquad\quad \underline{- 7 - 35 - 28} \\
\qquad\qquad\qquad\qquad\quad 14 + 63 + 21 \\
\qquad\qquad\qquad\qquad\quad \underline{14 + 70 + 56} \\
\qquad\qquad\qquad\qquad\qquad - 7 - 35 - 28 \\
\qquad\qquad\qquad\qquad\qquad \underline{- 7 - 35 - 28}
\end{array}
$$

Hence quotient $= x^4 - x^3 - 7x^2 + 14x - 7$.

The student is recommended to apply this method to the examples in Exercise XVI.

SYNTHETIC DIVISION.

63. The following is a still shorter method of division, and is peculiarly applicable when the first coefficient of the divisor is unity. It is frequently called "Horner's Method;" after the name of its inventor.*

RULE.—*After properly arranging divisor and dividend, if the first coefficient of the divisor be not unity, divide both dividend and divisor by the first coefficient of the latter. Then set down the first term of the dividend for first term of the quotient.*

Arrange the divisor in a vertical column to the left of the dividend, and change the sign of every term in it except the first.

Multiply all the terms of the divisor, so changed, by the first term of the quotient, and arrange the products diagonally under the second and following vertical columns of the dividend.

Add the terms in the second column and the sum will be the second term of the quotient. Multiply the changed terms of the divisor by the second term of the quotient, and arrange the products under the third and following vertical columns of the dividend.

Continue this process until the remaining vertical columns added give zero for sum, or until, in other cases, the division is carried as far as desired.

NOTE.—It is usual in synthetic division to perform the work by detached coefficients, remembering to place 0s for the absent terms in both divisor and dividend.

Ex. 1. Divide $a^6 - 3a^4x^2 + 3a^2x^4 - x^6$ by $a^3 - 3a^2x + 3ax^2 - x^3$.

OPERATION.

$$\begin{array}{r|lllllll}
1 & 1 & +0 & -3 & +0 & +3 & +0 & -1 \\
+3 & & 3 & +9 & +9 & +3 & & \\
-3 & & & -3 & -9 & -9 & -3 & \\
+1 & & & & +1 & +3 & +3 & +1 \\
\hline
\text{Quot.} = & 1 & +3 & +3 & +1 & +0 & +0 & +0
\end{array} = a^3 + 3a^2x + 3ax^2 + x^3$$

* Synthetic division demands the attention of the student not only on account of its brevity and elegance, but also for its great value in many of the higher departments of research, such as in obtaining factors preparatory to the integration of finite differences, in constructing a recurring series, in the treatment of reciprocal equations, &c.

EXPLANATION.—Using only the coefficients we write a 0 for each absent term, i. e., for the terms involving a^5x, a^3x^3, and ax.

The first coef. of the divisor being unity, the first step of the rule is not required.

We set down the divisor vertically on the right of the dividend, and change all its signs except the first.

We place the first term of the dividend for first term of quotient.

We multiply the changed terms of the divisor by the first terms of the quotient, and arrange the products, 3, −3, and 1, diagonally as represented, so that the first is under the second term of the dividend, and so that each is horizontally opposite that term of the divisor from which it was obtained.

We add the second column, and get $+3$ for the second term of the quotient.

We multiply the changed terms of divisor by this $+3$, and arrange the products $+9$, -9, and $+3$, diagonally, as represented.

We add the third column, and thus get $+3$ for the third term of the quotient, and so on.

Lastly we attach the proper literal part to each term.

Ex. 2. Divide $6a^4 - a^3 + 2a^2 + 13a + 4$ by $2a^2 - 3a + 4$.

OPERATION.

$$
\begin{array}{r|l}
2 - 3 + 4\ ) & 6 - 1 + 2 + 13 + 4 \\
1 & 3 - \tfrac{1}{2} + 1 + 6\tfrac{1}{2} + 2 \\
+\ 1\tfrac{1}{2} & \quad + 4\tfrac{1}{2} + 6 + 1\tfrac{1}{2} \\
-\ 2 & \qquad\quad - 6 - 8\ \ - 2 \\
\hline
\text{Quot.} = & 3 + 4 + 1 + 0\ \ + 0 = 3a^2 + 4a + 1.
\end{array}
$$

EXPLANATION.—Here, as the first coefficient of the divisor is not unity, we divide both divisor and dividend by 2, the first coef. of the former. The rest of the process is similar to that in last example.

Ex. 3. Divide $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 7ax^4 - 5x^5$ by $a^2 - 2ax + x^2$.

$$
\begin{array}{r|r|l}
1 & 1 - 5 + 10 - 10 & + 7 - 5 \\
+\ 2 & + 2 - \ 6 + \ 6 & - 2 \\
-\ 1 & - \ 1 + \ 3 & - 3 + 1 \\
\hline
\text{Quot.} = & 1 - 3 + \ 3 - \ 1 & + 2 - 4 = a^3 - 3a^2x + 3ax^2 - x^3 + \dfrac{2ax^4 - 4x^5}{a^2 - 2ax + x^2}.
\end{array}
$$

EXPLANATION.—The vertical line is drawn in order to show where the remainder commences, and it will be observed that this is one less than as many columns from the extreme right as there are terms in the divisor.

The student is recommended to apply this method to the examples in Exercise XVI.

SECTION III.

THEOREMS* AND FACTORING.

64. The following theorems should be thoroughly mastered by the pupil :—

65. THEOREM I.—*Zero divided by any given quantity gives zero for quotient.*

DEMONSTRATION.—The divisor × quotient must = dividend, and consequently the smaller the dividend becomes, the divisor remaining unchanged, the smaller must the quotient be. Hence when the dividend becomes less than any assignable quantity, i. e., = 0, the quotient also becomes = 0, that is $0 \div a = 0$.

66. THEOREM II.—*A finite quantity divided by zero gives an infinitely large quantity for quotient.*

DEMONSTRATION.—A finite quantity divided by itself gives unity for quotient, and as the divisor is decreased in magnitude (the dividend remaining unaltered), the quotient increases. Hence when the divisor becomes infinitely small, i. e. = 0, the quotient becomes infinitely large, i. e. = ∞. Therefore $a \div 0 = \infty$.

67. THEOREM III.—*A finite quantity divided by a quantity infinitely large, gives a quotient infinitely small, or in other words gives zero for quotient.*

DEMONSTRATION.—Since the divisor × quotient = dividend, it is evident that (the dividend remaining unchanged), the larger the divisor the smaller must be the other factor or quotient. When then the divisor becomes infinitely great the quotient must become infinitely small. Hence $a \div \infty = 0$.

68. THEOREM IV.—*Zero divided by zero gives any quantity whatever for quotient.*

DEMONSTRATION.—Since the divisor × quotient = dividend, and the dividend and divisor are both zero, it follows that the quotient may be any quantity whatever, or in other words, $0 \div 0 = a$, because $0 \times a = 0$.

* An algebraic theorem is an algebraic property required to be demonstrated.

69. THEOREM V.—*The zero power of any quantity is equal to unity.*

DEMONSTRATION.—Since one power of a quantity is divided by another power of the same quantity by subtracting the exponent of the divisor from that of the dividend, it follows that $a \div a = a^{1-1} = a^0$; but any quantity divided by itself equals unity, hence $a \div a = 1$. Since then $a \div a = a^0$ and also $= 1$, it is evident that $a^0 = 1$.

Cor. Similarly it may be shown that $\frac{1}{a}$ and a^{-1} are equivalent expressions:—for $\dfrac{1}{a} = \dfrac{a^0}{a} = a^{0-1} = a^{-1}$.

NOTE.—It follows from the foregoing theorems that a being any finite quantity whatever,

0, $\dfrac{0}{a}$ and $\dfrac{a}{\infty}$ are equivalent symbols, each representing no quantity, or the absence of quantity, or a quantity less than any assignable quantity.

$\dfrac{a}{0}$ and ∞ are equivalent symbols, each representing a quantity greater than any assignable quantity. Hence also, zero and infinity are the reciprocals of each other.

a^0, and $\dfrac{a}{a}$ and 1 are equivalent symbols, each representing unity.

$\dfrac{0}{0}$ is a symbol of indetermination, i. e. is employed to designate a quantity which admits of an infinite number of values, or, as we shall see hereafter, a quantity whose value depends upon its origin.

70. THEOREM VI.—*The square of the sum of any two quantities is equal to the sum of the squares of the two quantities together with twice their product.*

DEMONSTRATION.—Let a and b be the two quantities; then $a+b$ = their sum, and $(a+b)^2$ = the square of their sum.

Now $(a+b)^2 = (a+b)\,(a+b) = a^2 + 2ab + b^2$.

71. THEOREM VII.—*The square of the difference of any two quantities is equal to the sum of the squares of the two quantities diminished by twice their product.*

DEMONSTRATION.—Let a and b be the two quantities; then $a-b$ = their difference, and $(a-b)^2$ = the square of their difference.

Now $(a-b)^2 = (a-b)\,(a-b) = a^2 - 2ab + b^2$.

72. THEOREM VIII.—*The product of the sum of any two quantities by the difference of the same two quantities is equal to the difference of the squares of the two quantities.*

DEMONSTRATION.—Let a and b be the two quantities, a being the greater; then $(a+b)$ = the sum, and $(a-b)$ = the difference of the quantities, and

$$(a+b)\ (a-b) = a^2 - b^2 = \text{diff. of their squares.}$$

73. THEOREM IX.—*The product of two binomials having the same quantity for first term but their second terms unlike, is equal to the square of the first term together with the product of the two second terms and also the product of the first term by the sum of the two second terms.*

DEMONSTRATION.—Let $(x+a)$ and $(x-b)$ be the two binomials, then by actual multiplication $(x+a)\ (x-b) = x^2 + (a-b)x - ab$.

Similarly if $(x-a)$ and $(x-b)$ are the two binomials, their product will be $x^2 + (-a-b)x + ab = x^2 - (a+b)x + ab$.

74. THEOREM X.—*The difference of the* n^th^ *powers of two quantities is always divisible by the difference of the simple powers of the same two quantities whether the exponent* n *be an odd number or an even number.*

DEMONSTRATION. We are to show that the two quantities being a and x, and the difference of their n^{th} powers being $a^n - x^n$, then $a^n - x^n$ is divisible by $a - x$ whether n be an odd number or an even number.

$$\frac{a^n - x^n}{a-x} = a^{n-1} + \frac{a^{n-1}x - x^n}{a-x} = a^{n-1} + \frac{x(a^{n-1} - x^{n-1})}{a-x}$$

Now it is evident that when $a^{n-1} - x^{n-1}$ is divisible by $a-x$ then $a^n - x^n$ must also be divisible by $a-x$.

But when $n = 2$, $n-1 = 1$, and it is manifest that $a-x$ is divisible by $a-x$, therefore $a^2 - x^2$ is divisible by $a-x$.

Again if $n = 3$, $n-1 = 2$, and since $a^2 - x^2$ is divisible by $a-x$, then also $a^3 - x^3$ is divisible by $a-x$, and hence also $a^4 - x^4$ is divisible by $a-x$, and hence also $a^5 - x^5$ and so on. Therefore $a^n - x^n$ is exactly divisible by $a-x$, whether n be an odd or an even number

75. THEOREM XI.—*The sum of the* n^th^ *powers of any two quantities is not divisible by the difference of the quantities whether* n *be an odd or an even number.*

DEMONSTRATION. $$\frac{a^n + x^n}{a - x} = a^{n-1} + \frac{x(a^{n-1} + x^{n-1})}{a - x}$$

Now $a^n + x^n$ is div. by $a - x$ only when $a^{n-1} + x^{n-1}$ is div. by $a - x$.

Taking $n = 2$, $n - 1 = 1$, and $a^{n-1} + x^{n-1} = a + x$, which is evidently not div. by $a - x$, and therefore $a^2 + x^2$ is not div. by $a - x$.

But when $n = 3$, $n - 1 = 2$, and since $a^2 + x^2$ is not div. by $a - x$, therefore $a^3 + x^3$ is not div. by $a - x$.

But when $n = 4$, $n - 1 = 3$, and since $a^3 + x^3$ is not div. by $a - x$, therefore $a^4 + x^4$ is not div. by $a - x$.

And therefore $a^5 + x^5$ is not div. by $a - x$, and therefore $a^6 + x^6$ is not div. by $a - x$, and so on.

Therefore whether n be even or odd, $a^n + x^n$ is not div. by $a - x$.

76. THEOREM XII.—*The difference of the* n^th^ *powers of any two quantities is not divisible by the sum of the quantities when* n *is an odd number.*

DEMONSTRATION. $$\frac{a^n - x^n}{a + x} = a^{n-1} - a^{n-2}x + \frac{x^2(a^{n-2} - x^{n-2})}{a + x}$$

Now $a^n - x^n$ is div. by $a + x$ only when $a^{n-2} - x^{n-2}$ is div. by $a + x$.

Taking $n = 3$, $n - 2 = 1$, and $a^{n-2} - x^{n-2} = a - x$, which is evidently not div. by $a + x$, and therefore $a^3 - x^3$ is not div. by $a + x$.

But when $n = 5$, $n - 2 = 3$, and since $a^3 - x^3$ is not div. by $a + x$, therefore also $a^5 - x^5$ is not div. by $a + x$.

But when $n = 7$, $n - 2 = 5$, and since $a^5 - x^5$ is not div. by $a + x$, therefore also $a^7 - x^7$ is not div. by $a + x$, and so on.

Therefore when n is an odd number, $a^n - x^n$ is not div. by $a + x$.

77. THEOREM XIII.—*The sum of the* n^th^ *powers of any two quantities is not divisible by the sum of the quantities when* n *is an even number.*

DEMONSTRATION. $$\frac{a^n + x^n}{a + x} = a^{n-1} - \frac{x(a^{n-1} - x^{n-1})}{a + x}$$

Now in order that $a^n + x^n$ shall be div. by $a + x$, $a^{n-1} - x^{n-1}$ must be div. by $a + x$.

When n = an even number, $n - 1$ must = an odd number; and we have shown (Theor. XII.) that the difference of the odd powers of two quantities is not div. by the sum of the quantities. Therefore when n is an even number, $a^{n-1} - x^{n-1}$ is not div. by $a + x$, and therefore $a^n + x^n$ is not div. by $a + x$ when n is an even number.

78. THEOREM XIV.—*The difference of the* n^th^ *powers of any two quantities is exactly divisible by the sum of the quantities when* n *is an even number.*

DEMONSTRATION. $$\frac{a^n - x^n}{a+x} = a^{n-1} - \frac{x(a^{n-1} + x^{n-1})}{a + x}$$

Now when $a^{n-1} + x^{n-1}$ is div. by $a + x$, then also $a^n - x^n$ is div. by $a + x$.

But when $n = 2$, $n - 1 = 1$, and $a + x$ is evidently div. by $a + x$, therefore $a^2 - x^2$ is div. by $a + x$.

And by first step of next theorem $a^3 + x^3$ is div. by $a + x$, and therefore also $a^4 - x^4$ is div. by $a + x$, and so on.

Therefore $a^n + x^n$ is divisible by $a + x$, when n is an even number.

NOTE.—The several steps of this and of the following demonstration mutually depend upon one another. Thus, the 1st step of the following depends on the 1st step of this; 2nd step of this on 1st step of following; 2nd step of following on 2nd step of this; 3rd step of this on 2nd step of following; and so on.

79. THEOREM XV.—*The sum of the* n^th^ *powers of any two quantities is divisible by the sum of the quantities when* n *is an odd number.*

DEMONSTRATION. $$\frac{a^n + x^n}{a+x} = a^{n-1} + \frac{x(a^{n-1} - x^{n-1})}{a + x}$$

Now $a^n + x^n$ is exactly div. by $a + x$ when $a^{n-1} - x^{n-1}$ is div. by $a + x$.

But when n = an odd number, $n - 1$ must = an even number, and $a^{n-1} - x^{n-1}$ expresses the *difference* of two even powers, and since (1st step of Theorem XIV.) $a^2 - x^2$ is divisible by $a + x$, therefore also $a^3 + x^3$ is divisible by $a + x$.

And since (2nd step of Theorem XIV.) $a^4 - x^4$ is divisible by $a+x$, therefore also $a^5 + x^5$ is divisible by $a+x$; and so on.

Therefore $a^n + x^n$ is div. by $a+x$ when n = an odd number.

80. The following is a recapitulation of the latter of these theorems:—

$a^n - x^n$ is div. by $a - x$ when n is odd.

$a^n - x^n$ is div. by $a - x$ when n is even.

$a^n + x^n$ is div. by $a + x$ when n is odd.

$a^n - x^n$ is div. by $a + x$ when n is even.

All other nth powers are indivisible by either $a + x$ or $a - x$.

ILLUSTRATIVE EXAMPLES.

THEOREM VI.

$(2x + 3y^2)^2 = (2x)^2 + 2(2x)(3y^2) + (3y^2)^2 = 4x^2 + 12xy^2 + 9y^4$.

$(2ax + 5yz)^2 = (2ax)^2 + 2(2ax)(5yz) + (5yz)^2 = 4a^2x^2 + 20axyz + 25y^2z^2$.

Conversely $x^2 + 2xy + y^2 = (x+y)(x+y)$; $a^2 + 4ax + 4x^2 = (a+2x)(a+2x)$; $9a^2 + 6axy + x^2y^2 = (3a + xy)(3a + xy)$; $4x^4 + 12x^2y + 9y^2 = (2x^2 + 3y)(2x^2 + 3y)$.

THEOREM VII.

$(m - 2x)^2 = m^2 - 2(m)(2x) + (2x)^2 = m^2 - 4mx + 4x^2$

$(4ab - 3x^2y)^2 = (4ab)^2 - 2(4ab)(3x^2y) + (3x^2y)^2 = 16a^2b^2 - 24abx^2y + 9x^4y^2$.

Conversely $m^2 - 2my + y^2 = (m - y)(m - y)$; $4x^2y^2 - 4acxy + a^2c^2 = (2xy - ac)(2xy - ac)$.

THEOREM VIII.

$$(m - xy)(m + xy) = m^2 - (xy)^2 = m^2 - x^2y^2$$

$$(3a + 7y)(3a - 7y) = (3a)^2 - (7y)^2 = 9a^2 - 49y^2.$$

$(4a^2xy - 3a^3b)(4a^2xy + 3a^3b) = (4a^2xy)^2 - (3a^3b)^2 = 16a^4x^2y^2 - 9a^6b^2$.

Conversely $x^2 - 4y^2 = x^2 - (2y)^2 = (x + 2y)(x - 2y)$; $x^4y^4 - m^4b^2 = (x^2y^2)^2 - (m^2b)^2 = (x^2y^2 + m^2b)(x^2y^2 - m^2b)$.

$x^4 - a^4 = (x^2 + a^2)(x^2 - a^2) = (x^2 + a^2)(x + a)(x - a)$.

$m^{16} - a^{16}b^{16} = (m^8 + a^8b^8)(a^8 - a^8b^8) = (m^8 + a^8b^8)(m^4 + a^4b^4)(m^4 - a^4b^4) = (m^8 + a^8b^8)(m^4 + a^4b^4)(m^2 + a^2b^2)(m^2 - a^2b^2)$

$$= (m^8 + a^8b^8)(m^4 + a^4b^4)(m^2 + a^2b^2)(m + ab)(m - ab).$$

THEOREM IX.

$$(x-7)(x+9) = x^2+(9-7)x-63 = x^2+2x-63.$$
$$(x-3)(x-7) = x^2-(3+7)x+21 = x^2-10x+21.$$

Conversely. Find the factors of $x^2+14x+33$. Here since 14 is the sum and 33 the product of the two last terms, we seek to find by inspection what numbers added will make 14 and multiplied together will make 33. Evidently 11 and 3.

Therefore $x^2+14x+33 = (x+11)(x+3)$

$x^2+x-42 = (x+7)(x-6)$ $\because$ $7+(-6)=1$ and $7\times-6=-42$.

$x^2-9x+20=(x-5)(x-4)$ $\because$ $-5+(-4)=-9$ and $-5\times-4=+20$.

$x^2-x-156=(x-13)(x+12)$ $\because$ $-13+12=-1$ and $-13\times12=-156$.

THEOREMS X., XIV., and XV.—By actual division,

$$\frac{a^4-x^4}{a-x}=a^3+a^2x+ax^2+x^3\,;\ \frac{a^4-x^4}{a+x}=a^3-a^2x+ax^2-x^3.$$

$$\frac{a^5-x^5}{a-x}=a^4+a^3x+a^2x^2+ax^3+x^4\,;\ \frac{a^5+x^5}{a+x}=a^4-a^3x+a^2x^2-ax^3+x^4.$$

81. In order to be enabled to write these and similar quotients without actually dividing, observe the following points :—

I. The number of terms in the quotient always = the exponent of a in the dividend ÷ exponent of a in the divisor.

II. The coef. of each term of the quotient is unity.

III. The exponent of a decreases and that of x increases in the several terms of the quotient, by unity, or more generally by the exponent of the corresponding term of the divisor.

IV. When the connecting sign of the divisor is *minus*, all the signs of the quotient are +, but when the connecting sign of the divisor is *plus*, the signs of the quotient are + and − alternately.

V. The sum of the exponents of each term = the difference between the exponent of a in the dividend and that of a in the divisor.

EXERCISE XVII.

Find by inspection the value of :—

1. $(a-3y)^2$; $(3a+2x)^2$; $(3xy-7)^2$; $(2ax^2-3x)^2$; $(2a+3axy^2)^2$.

2. $(a-3x)(a+3x)$; $(2a+3y)(2a-3y)$; $(3ab-xy)(xy+3ab)$; $(2m^2-3xy^3)(2m^2+3xy^3)$.

3. $(3a-2xy)(2xy+3a)$; $(2a-7)(7+2a)$; $(x+3)(3-x)$; $(2+5ay)^2$; $(3a-4x^2y^3)^2$.

4. $(x-6)(x+11)$; $(3a-2)(3a+5)$; $(x-4)(x-9)$; $(x+3)(x-7)$; $(x-2)(x-1)$.

5. $(a^7-x^7)\div(a-x)$; $(a^6-x^6)\div(a+x)$; $(m^5+a^5)\div(m+a)$; $(c^4+x^4)\div(c+x)$.

6. $(a^{11}+x^{11}y^{11})\div(a+xy)$; $(a^9m^9-r^9)\div(am-r)$; $(a^8+m^8s^8)\div(a-ms)$; $(a^4-y^4z^4)\div(a-yz)$.

7. $(x^2+9x+20)\div(x+5)$; $(x^2+7x-8)\div(x-1)$; $(6x^2+5x-4)\div(3x+4)$; $(6a^4x^2+a^3x-a^2)\div(2ax+1)$.

---

82. Theorem VIII. may sometimes enable us to find without *actual multiplication* the product of two trinomials or quadrinomials, i. e., when we can write one of them as the sum of two quantities and the other as the difference of the same two quantities.

Ex. 1. $(a-x+y)(a-x-y)=\{(a-x)+y\}\{(a-x)-y\}=(a-x)^2-y^2=a^2-2ax+x^2-y^2$.

Ex. 2. $(2x-3y-2z)(2x+3y-2z)=\{(2x-2z)-3y\}\{(2x-2z)+3y\}=(2x-2z)^2-(3y)^2=4x^2-8xz+4z^2-9y^2$.

Ex. 3. $(a-2b+3c)(a+2b-3c)=\{a-(2b-3c)\}\{a+(2b-3c)\}=a^2-(2b-3c)^2=a^2-(4b^2-12bc+9c^2)=a^2-4b^2+12bc-9c^2$.

Ex. 4. $(a+2b+3c-d)(a-2b+3c+d)$
$=\{(a+3c)+(2b-d)\}\{(a+3c)-(2b-d)\}=(a+3c)^2-(2b-d)^2$
$=a^2+6ac+9c^2-(4b^2-4bd+d^2)=a^2+6ac+9c^2-4b^2+4bd-d^2$.

EXERCISE XVIII.

Find the value of :—

1. $(a-b+c)(a-b-c)$; $(a-b+c)(a+b-c)$; $(a+b+c)(a-b-c)$.

2. $(3a-2c+4)(4-3a+2c)$; $(2a-x+3m^2)(2a+x-3m^2)$; $(2a-3y+2xy)(3y-2a+2xy)$.

3. $(2a-3c+2x-3y)(3y-2x-3c+2a)$; $(a+2c+4m+3d)(a+3d-2c-4m)$.

4. $(3a-m^2-2+xy)(2-m^2+3a-xy)$; $(1+2a^2-3x^4+y^2)(2a^2-1-y^2-3x^2)$.

Simplify the following expressions, i. e. perform the operations indicated and reduce the result to its simplest form :—

5. $(3a-2b)(2a+3b)-(2a-4b)^2-4(3-a)(a+3)-4(2a-b)^2$.

6. $(4a-3xy)(3xy-4a)+3(2a+xy)^2-7(3a+xy)(xy-3a)+4(2a-3xy)^2$.

7. $(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\ldots.8$ terms.

8. $(a-xy)(a+xy)(a^2+x^2y^2)(a^4+x^4y^4)\ldots.$ to n terms.*

---

83. Although we have seen (Theor. XI and XII) that the sum of the even powers of any two quantities is not divisible either by the sum or the difference of the quantities, it sometimes happens that we can resolve the sum of two even powers into its component factors. This occurs whenever the exponent n contains an odd factor, as for example when it is 6, or 10, or 12, or 14, &c.

Ex. 1.—Resolve $a^3-x^3y^3$ into its elementary factors.

Theor. X. $a^3-x^3y^3=a^3-(xy)^3=(a-xy)(a^2+axy+x^2y^2)$.

Ex. 2.—Resolve a^5-m^5 into its elementary factors.

a^5-m^5 is divisible by $a-m$, and therefore its factors are $(a-m)(a^4+a^3m+a^2m^2+am^3+m^4)$.

Ex. 3.—What are the factors of x^7+y^{14}?

$x^7+y^{14}=x^7+(y^2)^7=(x+y^2)(x^6-x^5y^2+x^4y^4-x^3y^6+x^2y^8-x\,y^{10}+y^{12})$.

Observe here the exponents of x in the second factor decrease by the subtraction of that of x in the first factor, while the exponents of y in the second factor increase by the addition of that of y in the first factor.

Ex. 4.—What are the factors of $a^{16}-m^{16}c^{16}$?

By Theor. VIII. $a^{16}-(mc)^{16}=\{a^8+(mc)^8\}\{a^8-(mc)^8\}$ and $a^8-(mc)^8=\{a^4+(mc)^4\}\{a^4-(mc)^4\}$; and so on. Therefore $a^{16}-m^{16}c^{16}=(a^8+m^8c^8)(a^4+m^4c^4)(a^2+m^2c^2)(a+mc)(a-mc)$.

* Ascertain by inspection what power of 2 expresses the exponent of each term of the product of the first two of these factors, then of three, and hence of n factors.

Ex. 5.—What are the factors of $32x^5 + 243y^5$?

$32x^5 + 243y^5 = (2x)^5 + (3y)^5 = (2x + 3y)\{(2x)^4 - (2x)^3(3y) + (2x)^2(3y)^2 - (2x)(3y)^3 + (3y)^4\} = (2x+3y)(16x^4 - 24x^3y + 36x^2y^2 - 54xy^3 + 81y^4)$.

Ex. 6.—Resolve $a^{12} + m^{12}$ into its two elementary factors.

$a^{12} + m^{12} = (a^4)^3 + (m^4)^3$, and since the sum of the cubes of two quantities is divisible by the sum of the quantities, $(a^4)^3 + (m^4)^3 = (a^4 + m^4)(a^8 - a^4m^4 + m^8)$.

Ex. 7.—Resolve $a^{20} - x^{20}$ into six elementary factors.

$a^{20} - x^{20} = (a^{10} + x^{10})(a^5 + x^5)(a^5 - x^5)$.

$a^{10} + x^{10} = (a^2)^5 + (x^2)^5 = (a^2 + x^2)(a^8 - a^6x^2 + a^4x^4 - a^2x^6 + x^8)$, and resolving $(a^5 + x^5)$ and $a^5 - x^5$ into their factors, we find that $a^{20} - x^{20} = (a^2 + x^2)(a^8 - a^6x^2 + a^4x^4 - a^2x^6 + x^8)(a + x)(a^4 - a^3x + a^2x^2 - ax^3 + x^4)(a - x)(a^4 + a^3x + a^2x^2 + ax^3 + x^4)$.

Ex. 8.—Resolve $m^{54} - z^{54}$ into eight elementary factors.

$m^{54} - z^{54} = (m^{27} + z^{27})(m^{27} - z^{27})$.

$m^{27} + z^{27} = (m^9)^3 + (z^9)^3 = (m^9 + z^9)(m^{18} - m^9z^9 + z^{18})$ and $m^9 + z^9 = (m^3)^3 + (z^3)^3 = (m^3 + z^3)(m^6 - m^3z^3 + z^6)$ and

$m^3 + z^3 = (m + z)(m^2 - mz + z^2)$.

Therefore $m^{27} + z^{27} = (m^{18} - m^9z^9 + z^{18})(m^6 - m^3z^3 + z^6)(m^2 - mz + z^2)(m + z)$.

And similarly $m^{27} - z^{27} = (m^{18} + m^9z^9 + z^{18})(m^6 + m^3z^3 + z^6)(m^2 + mz + z^2)(m - z)$.

Therefore $m^{54} - z^{54}$ = the above eight factors.

EXERCISE XIX.

Resolve into elementary factors :—

1. $a^3 - m^3$; 2. $a^5 + c^5$; 3. $a^4 + x^4$; 4. $a^6 - b^6$;
5. $a^9 - x^9$; 6. $a^{11} - b^{11}$; 7. $a^4 - m^4x^4$; 8. $32a^5 + x^5$;
9. $81 - 16c^4$; 10. $243m^5 - 32c^5$; 11. $a^{22} + x^{22}$; 12. $a^{20} + m^{20}$;
13. $c^{24} + x^{24}$; 14. $x^{30} + m^{30}$; 15. $a^{48} - c^{48}$; 16. $a^{96} + m^{96}$;
17. $a^{108} - c^{108}$; 18. $m^{144} + c^{144}$; 19. $a^{14} + m^{14}$; 20. $(am)^{81} - p^{81}$.

EXERCISE XX.

MISCELLANEOUS EXAMPLES.

1. Simplify $a - x - \{-(-a) - x\} - \{-(-\{-a - (-\{-(-x - a) - a\} - x) - a\} - a)\}$

2. Simplify $3(a - x)(a + x) - 2(a - 2x)^2 - (3a - 2x)(2x - 3a) - 4(3x - a)(a + 3x)$.

3. Add together $\sqrt{3} + 2\sqrt{6} + 3\sqrt{5} - \sqrt{x}$; $2\sqrt{3} - 3\sqrt{5} - 4ax^2 - \sqrt{x}$, $2\sqrt{5} - 3\sqrt{x} + a^2x^2 - \sqrt{2}$; and $4\sqrt{6} - 3a^2x - 3\sqrt{x}$.

4. Multiply $a^m + x^{p+q}$ by $a^c - x^{m-p}$.

5. Divide $a^n - x^n$ by $a + x$ to 5 terms.

6. What are the factors of $x^2 - 14x - 51$?

7. Divide 1 by $1 - 1$, and express the value of the quotient.

8. Resolve $a^{18} - x^{18}$ into its six elementary factors.

9. Divide $a^4x^4m^2 - 4a^2m^2x^3p + 4p^2m^2x^2$ into its factors.

10. If $a = 2$, $b = 3$, $c = 4$, $d = 1$, and $m = 0$, find the value of

$$\frac{\sqrt{cd(ab + bd)}}{bc - m} + \sqrt[3]{a^2b^2cdm} - \frac{(\{a(b + c) - d\}^2 + ab) - \{bc(b + c) + 1\}}{cdm + \sqrt{c(bc + d)} - b - (a + b + d)}$$

11. Multiply by detached coefficients $x^4 + 2x^3 + 3x^2 + 2x + 1$ by $x^2 - 2x + 1$, and also $a^2 - 2ab - 3b^2$ by $a^3 + 2a^2b + 3ab^2 + 4b^3$.

12. Divide synthetically $x^4 - a^2x^2 + bx^2 - cx^2 + abx + acx - bc$ by $ax + x^2 - c$.

13. Resolve $a^{64} - m^{64}$ into its elementary factors.

14. Find by inspection the value of $(a^2 + c^2)(a + c)(a - c)(a^{20} - a^{18}c^2 + a^{16}c^4 - a^{14}c^6 + a^{12}c^8 - a^{10}c^{10} + a^8c^{12} - a^6c^{14} + a^4c^{16} - a^2c^{18} + c^{20})(a^{10} + a^9c + a^8c^2 + a^7c^3 + a^6c^4 + a^5c^5 + a^4c^6 + a^3c^7 + a^2c^8 + ac^9 + c^{10})(a^{10} - a^9c + a^8c^2 - a^7c^3 + a^6c^4 - a^5c^5 + a^4c^6 - a^3c^7 + a^2c^8 - ac^9 + c^{10})$.

15. If $a = \frac{1}{2}$, and $a + b + c = a + b = 0$, find the value of

$$(b^2 - c^2)\{b^2 + c^2 - b(a - c)\}$$

16. Simplify $a^3 - b^3 - 3ab(a - b) + 3ab(a + b) + a^3 + b^3$.

17. Simplify $a^2 - m^2 + 3(a - m)^2 - 2(2a - 3m)(3m + 2a) - 2m(5m + 3a) + 6(a^2 - m^2) + 2m(5a - 2m)$.

18. If $m = a + b + c$, prove that

$$m(m - 2a)(m - 2b) + m(m - 2b)(m - 2c) + m(m - 2c)(m - 2a) = 8abc + (m - 2a)(m - 2b)(m - 2c).$$

SECTION IV.

GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE.

GREATEST COMMON MEASURE.

84. The greatest common measure of two or more algebraic quantities is the letter or quantity of highest dimensions that will go into each of them without a remainder.

Thus, the greatest common measure (G. C. M.) of $4a^2xy$ and $6a^2xz^2$ is $2a^2x$; the G. C. M. of $3x^3y - 21x^2y$ and $2abx - 14ab$ is $x - 7$ or $7 - x$.

85.—The words *greater* and *less* are not generally applicable to algebraic expressions, unless when specific numerical values have been assigned to all the letters which occur in them. Thus, $x - 7$ is greater or less than $7 - x$, according as we assign different values to x. On this account the term Greatest Common Measure is incorrect as employed in Algebra, and, as we merely use the expression to indicate the common divisor of *highest dimensions*, it would be more accurate to call it the *highest common measure*.

86. THEOREM I.—*If a quantity measure another quantity it will also measure any multiple of that quantity.*

DEMONSTRATION.—We are to show that if m measure a then it will also measure ta, any multiple of a.

Let m be contained n times in a. Then $a = nm$, and $ta = tnm$. Now m evidently measures tnm, therefore it also measures its equal ta.

87. THEOREM II.—*If one quantity measure two other quantities then it will also measure the sum or difference of any multiples of those two quantities.*

DEMONSTRATION.—We are to show that if m measure a and also b, it will likewise measure $na \pm pb$.

Since m measures a and b by hypothesis, it also (Theor. I) measures na and pb Let m be contained t times in na and s times in pb; then $na = tm$ and $pb = sm$. Therefore $na \pm pb = tm \pm sm = (t \pm s)m$. That is, m is contained $(t \pm s)$ times in $na \pm pb$ and is therefore a measure of $na \pm pb$.

88. The G. C. M. of two or more quantities can often be found by inspection or by the following:—

RULE.—*Resolve each of the quantities into its component factors: then the product of those factors common to all the given quantities will be their G. C. M.*

Ex. 1. What is the G. C. M. of $49a^2b^2c^4$ and $63a^5b^3c^3$?

$49a^2b^2c^4 = 7a^2b^2c^3 \times 7c$ and $63a^5b^3c^3 = 7a^2b^2c^3 \times 9a^3b$, whence it is evident that the G. C. M. required is $7a^2b^2c^3$.

Ex. 2. The G. C. M. of $m^2(a^2-m^2)^2$ and $(a^2m+am^2)^3$;

that is, of $m^2(a^2-m^2)\ (a^2-m^2)$ and $\{am(a+m)\}^3$.

that is, of $m^2(a+m)(a-m)(a+m)(a-m)$ and $a^3m^3\ (a+m)(a+m)\ (a+m)$;

that is, of $m^2(a+m)^2\ (a-m)^2$ and $m^2(a+m)^2\ (a+m)\ a^3m$ is $m^2(a+m)^2$.

Ex. 3.—The G. C. M. of $15(x^2-2ax-3a^2)$ and $35(x^3+a^3)$.

that is, of $5\times 3\ (x+a)\ (x-3a)$ and $5\times 7\ (x+a)\ (x^2-ax+a^2)$;

that is, of $5(x+a)\times 3(x-3a)$ and $5(x+a)\times 7\ (x^2-ax+a^2)$ is $5(x+a)$.

EXERCISE XXI.

Find by factoring the G. C. M. of

1. $18ab^3m$ and $24a^2b^2m^3$.
2. $21a^4m^2$, $18a^3m^3$ and $15a^2m^4$.
3. $8a^2x^2y+17amxy-3a^2m^2x^2y$ and $5xy+3axy-14a^2x^2y$.
4. $x^2+2x-mx^2-2mx$ and $x^2+4x+4+ax+2a$.
5. $3a^2\ (a^2-x^2)$ and $4a^2x^2\ (a-x)^2$.
6. $3m^3(a^3-m^3)\ (a+m)$, $4m(a^2m-m^3)^2$ and $4m^2(a^2-m^2)\ (a-m)$.
7. $x^2-4x-21$, $x^2-12x+35$ and $x^2+5x-84$.
8. $(ax-a)^2$ and $a^2(x^2-3x+2)$.
9. $x+3x-4$, x^2-2x+1 and x^2-1.

89. To find the G. C. M. of two polynomials:—

RULE.

I. *Strike out the greatest monomial factor (if there be any) which is common to all the terms of both polynomials, and reserve it.*

II. *Reject from each of the polynomials any remaining monomial factor that may be common to all its terms.*

III. *Arrange the resulting polynomials as for division, i.e., according to the powers of the same letter of reference, and make that one the divisor whose first term is of lower, or of not higher dimensions, as to the letter of reference, than the first term of the other.*

IV. *Multiply (if necessary) the dividend by the least monomial that will render its first term exactly divisible by the first term of the divisor.*

V. *Divide the dividend by the divisor and continue the division until the highest exponent of the letter of reference in the remainder is less than the exponent of the letter of reference in the first term of the divisor, observing that if the coef. of the first term of any partial rem. should happen not to be divisible by the coef. of the first term of the divisor, in order to avoid fractions, the rem. is to be multiplied by such a number as will render the coef. of its first term exactly divisible by the coef. of the first term of the divisor.*

VI. *Reject from the remainder its greatest monomial factor, and if its first term is negative, change all its signs: consider the result as constituting a new divisor and the former divisor a new dividend: proceed as before, and continue the operation until there is no remainder.*

VII. *Multiply the last divisor by the reserved monomial, if any, and the product will be the G. C. M. of the given polynomials.*

PROOF OF RULE.—The G. C. M. of two quantities is evidently the product of all the factors common to both. Hence if we reject any monomial factor common to both (as we may do for the sake of convenience) we must still regard this factor as entering into the G. C. M., and therefore we reserve it.

II.—Since the G. C. M. of two quantities is the product of all the factors which are *common to both* quantities, it is evident that a factor which belongs only to one of the two cannot form a part of their G. C. M., and therefore we may, for the sake of abbreviating the work, reject as directed in II.

IV.—Having by II struck out every monomial that is a factor of either of the quantities, it is evident that if we multiply the dividend by any monomial in order to make its first exactly divisible by the first term of the divisor, this monomial not being a factor of each of the terms of the divisor (though it is of the first term) cannot be a factor common to both dividend and divisor, and therefore cannot form part of their G. C. M.

III, V, VII.—Let the given polynomials whose G. C. M. is required be m^2na and m^2fb, where m^2, n and f are monomials. After striking out and reserving the common factor m^2, and rejecting from the remainders na and fb, the factors n and f which are not common to both; then the reduced polynomials whose G.C.M. is sought are a and b. Suppose these being properly arranged, the leading letter of b is of lower or not higher dimensions than that of a. Then divide and suppose $a \div b$ gives a quotient p with rem. c; also $b \div c$ gives quotient q and rem. d; also $c \div d$ gives quotient r and no rem. Then d is the G. C. M. of a and b.

```
b ) a ( p
    bp
    --
    c ) b ( q
        cq
        --
        d ) c ( r
            dr
            --
```

We shall first show that d is a common measure of a and b.

Because d measures c, since it goes into it without a remainder, therefore (Theor. I) it measures qc a multiple of c.

Because d measures d and also qc, therefore (Theor. II) it measures their sum, which is b.

Because d measures b it also measures pb, a multiple of b.

Because d measures pb and also c it measures their sum which is a.

Therefore d measures both b and a, and is a common measure of them.

Next we shall show that d being a common measure is the *greatest* common measure of a and b.

For if d be not the G. C. M. of a and b let there be a greater as d'.

Then because d' measures b it measures pb, a multiple of b.

Because d' measures a and also pb, it measures (Theor. II) their difference, which is c.

Because d' measures c it also measures qc, a multiple of c.

Because d' measures b and also qc it measures their difference, which is d.

Therefore d' measures d, that is, a greater quantity measures a less, which is absurd.

Therefore d' is not a common measure of a and b; and in like manner it may be shown that no quantity greater than d is a common measure of a and b. Therefore d is the G. C. M. of a and b.

V.—We may multiply any remainder by any number in order to make its first coef. exactly divisible by the first coef. of the divisor, because the G. C. M. of a and b is the same as the G. C. M. of any divisor b and rem. c. If now we multiply this rem. c by any monomial as f, the divisor b having no monomial factor, can have no factor in common with f, nor therefore any in common with fc but what it may have in common with c. That is, the G. C. M. of b and fc will be the same as the G. C. M. of b and c, and therefore the same as the G. C. M. of a and b.

VI.—We reject the monomial factor of the remainder before making it a divisor, because the former divisor, which has now become a dividend, contains no monomial factor, and therefore can contain no factor in common with the monomial rejected from what now becomes the divisor, and therefore the G. C. M. of the dividend (last divisor) and the unreduced divisor (i. e. last rem.) is the same as the G. C. M. of the dividend and divisor reduced as directed.

We can change all the signs of the divisor because this is equivalent merely to dividing it by -1.

Ex. 1. What is the G. C. M. of $x^2-10x+21$ and $x^2-2x-35$?

OPERATION.

$$
\begin{array}{l}
x^2-10x+21\ )\ x^2-\ \ 2x-35\ (\ 1 \\
\qquad\qquad\qquad\ \ \underline{x^2-10x+21} \\
\qquad\qquad\qquad\qquad\quad 8x-56=8(x-7)
\end{array}
$$

$$
\begin{array}{l}
x-7\ )\ x^2-10x+21\ (\ x-3 \\
\qquad\quad \underline{x^2-\ \ 7x} \\
\qquad\qquad\ -3x+21 \\
\qquad\qquad\ \underline{-3x+21}
\end{array}
$$

$\therefore$ G. C. M. $= x-7$.

EXPLANATION.—There is no monomial factor common to both, nor is there any monomial factor common to all the terms of either. Therefore we at once proceed to divide, x being taken as letter of reference; the first terms of the given quantities are of the same dimensions, and consequently it makes no difference which is taken as divisor.

After the first step of the division we obtain a remainder $8x-56$, and before using this for divisor we strike out its monomial factor 8. This gives us $x-7$ for 2nd divisor. We make the last divisor the new dividend, and finding that we now obtain no rem., we conclude that the G. C. M. is $x-7$.

Ex. 2.—Find the G. C. M. of $2a^4 + 3a^3x - 9a^2x^2$ and $6a^4x - 17a^3x^2 + 14a^2x^3 - 3ax^4$.

OPERATION.

$$\frac{6a^4x - 17a^3x^2 + 14a^2x^3 - 3ax^4}{2a^4 + 3a^3x - 9a^2x^2} = \frac{a \times x(6a^3 - 17a^2x + 14ax^2 - 3x^3)}{a \times a(2a^2 + 3ax - 9x^2)}$$

$$\begin{array}{r|l}
2a^2 + 3ax - 9x^2\ )\ 6a^3 - 17a^2x + 14ax^2 - 3x^3 & (\ 3a - 13x \\
6a^3 + 9a^2x - 27ax^2 & \\
\hline
-26a^2x + 41ax^2 - 3x^3 & \\
-26a^2x - 39ax^2 + 117x^3 & \\
\hline
80ax^2 - 120x^3 = 40x^2(2a - 3x) &
\end{array}$$

$$\begin{array}{r|l}
2a - 3x\ )\ 2a^2 + 3ax - 9x^2 & (\ a + 3x \\
2a^2 - 3ax & \\
\hline
6ax - 9x^2 & \\
6ax - 9x^2 & \\
\hline
\end{array}$$

G. C. M. of the reduced polynomials $= 2a - 3x$ and reserved common factor $= a$.

Therefore G. C. M. of given quantities $= a(2a - 3x)$.

EXPLANATION.—Here we strike out and reserve the monomial factor a, which is common to both quantities, and strike out and reject the monomial factor x of the second quantity and remaining monomial factor a of the first.

We select the divisor as shown in the margin, because a^2, its first term, is of lower dimensions than a^3, the first term of the other. Our first rem. is $80ax^2 - 120x^3$ from which we reject its greatest monomial factor $40x^2$, and this gives us $2a - 3x$ for a new divisor, the last divisor becoming the new dividend.

Ex. 3.—Find the G. C. M. of $6x^4 - x^3y - 3x^2y^2 + 3xy^3 - y^4$ and $9x^4 - 3x^3y - 2x^2y^2 + 3xy^3 - y^4$.

OPERATION.

$$6x^4 - x^3y - 3x^2y^2 + 3xy^3 - y^4\ )\ 9x^4 - 3x^3y - 2x^2y^2 + 3xy^3 - y^4\ (\ 3$$
$$2$$
$$18x^4 - 6x^3y - 4x^2y^2 + 6xy^3 - 2y^4$$
$$18x^4 - 3x^3y - 9x^2y^2 + 9xy^3 - 3y^4$$
$$- 3x^3y + 5x^2y^2 - 3xy^3 + y^4$$
$$= - y(3x^3 - 5x^2y + 3xy^2 - y^3)$$

$$3x^3 - 5x^2y + 3xy^2 - y^3\ )\ 6x^4 - x^3y - 3x^2y^2 + 3xy^3 - y^4\ (\ 2x + 3y$$
$$6x^4 - 10x^3y + 6x^2y^2 - 2xy^3$$
$$9x^3y - 9x^2y^2 + 5xy^3 - y^4$$
$$9x^3y - 15x^2y^2 + 9xy^3 - 3y^4$$
$$6x^2y^2 - 4xy^3 + 2y^4$$
$$= 2y^2(3x^2 - 2xy + y^2)$$

$$3x^2 - 2xy + y^2\ )\ 3x^3 - 5x^2y + 3xy^2 - y^3\ (\ x - y$$
$$3x^3 - 2x^2y + xy^2$$
$$- 3x^2y + 2xy^2 - y^3$$
$$- 3x^2y + 2xy^2 - y^3$$

Therefore G. C. M. $= 3x^2 - 2xy + y^2$

EXPLANATION.—Here, after seeing that the terms are properly arranged and that there is no monomial factor to reject, we multiply the dividend by 2 in order to make its first term exactly divisible by the first term of the divisor.

Before making the rem. a div. we cast out its monomial factor y and change all its signs, or, what amounts to the same thing, we cast out the monomial factor $-y$.

Before making the next rem. a new divisor we cast out its monomial factor $2y^2$.

EXERCISE XXII.

Find the G. C. M. of—

1. $x^2 - 5x - 14$ and $x^2 - x - 6$.
2. $x^4 - 8x^3 + 21x^2 - 20x + 4$ and $2x^3 - 12x^2 + 21x - 10$
3. $a^2 - ax - 7a + 7x$ and $a^3 - 3a + 3x - a^2x$.

4. $x^4 + x^3 - 12x$ and $x^3 + 4x^2 + 5x + 20$.

5. $a^2 - 3ab + 2b^2$ and $a^2 - ab - 2b^2$.

6. $a^3 - a^2b + 3ab^2 - 3b^3$ and $a^2 - 5ab + 4b^2$.

7. $30x^4 - 18x^3 + 94x^2 - 42x + 56$ and $60x^6 - 36x^5 + 48x^4 - 45x^3 + 42x^2 - 45x + 12$.

8. $6a^3b - 6a^2by - 2by^3 + 2aby^2$ and $12a^2b + 3by^2 - 15aby$.

9. $a^3 + 9a^2 + 27a - 98$ and $a^2 + 12a - 28$.

10. $8a^3b^2 - 24a^2b^3 + 24ab^4 - 8b^5$ and $12a^4 - 24a^3b + 12a^2b^2$.

11. $6a^5 + 20a^4 - 12a^3 - 48a^2 + 22a + 12$ and $a^6 + 4a^5 - 3a^4 - 16a^3 + 11a^2 + 12a - 9$.

12. $2a^3 - 2a^2b - 16ab^2 + 12b^3$ and $3a^4c - 9a^3bc - 24a^2b^2c + 54ab^3c - 24b^4c$.

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90. To find the G. C. M. of three quantities:—Find the G. C. M. of two of them, and then of this G. C. M. and the third quantity. To find the G. C. M. of four quantities:—Find the G. C. M. of any two of them, and then the G. C. M. of the other two, and lastly the G.C.M. of the two greatest common measures thus found.

LEAST COMMON MULTIPLE.

91. The Least Common Multiple (l. c. m.) of two or more algebraic quantities is the quantity of lowest dimensions, as to the letter or letters of reference, which exactly contains each of the given quantities.

NOTE.—Of course there is the same objection to the use of the word "*least*" here as to the word "*greatest*" in regard to common measures. It would be more correct to use the term *lowest common multiple.*

92. To find the l. c. m. of two or more algebraic quantities:—

RULE.—*Divide their product by their G. C. M.*

Or, Divide one of the given quantities by their G. C. M., and multiply the quotient and remaining quantity or quantities together for their l. c. m.

PROOF OF RULE.—Let it be required to find the l. c. m. of any two quantities a and b, and let m be the G. C. M. of these quantities.

Let $a = pm$ and $b = qm$, and m being the G. C. M. of a and b, it follows of course that p and q have no common factor. Then $pq =$ least quantity that contains both p and q, and $mpq =$ the least quantity that contains p, q, and m, and therefore $=$ the l. c. m. of a and b. Then l. c. m. $= pqm$ $= \frac{pm \times qm}{m} = \frac{a \times b}{m}$ or $= \frac{a}{m} \times b$ or $= a \times \frac{b}{m}$.

Ex. 1. Find the l. c. m. of $18a^2x^2y$ and $15ax^3y^2z$.

OPERATION.

G. C. M. of $18a^2x^2y$ and $15ax^3y^2z = 3ax^2y$.

Then $\frac{18a^2x^2y}{3ax^2y} \times 15ax^3y^2 = 6a \times 15ax^3y = 90a^2x^3y^2 =$ l. c. m.

Ex. 2. Find the l. c. m. of $a^3 + 3a^2 + 5a + 3$ and $a^3 + a^2 + a - 3$.

OPERATION.

G. C. M. of $a^3 + 3a^2 + 5a + 3$ and $a^3 + a^2 + a - 3 = a^2 + 2a + 3$.

$\frac{a^3 + 3a^2 + 5a + 3}{a^2 + 2a + 3} = a + 1$ and $(a^3 + a^2 + a - 3) \times (a + 1) = a^4 + 2a^3$ $+ 2a^2 - 2a - 3 =$ l. c. m.

93. *Very frequently the l. c. m. can be most easily obtained by resolving all the given quantities into their prime factors, and multiplying together the highest powers of all the factors that occur in order to form the l. c. m.*

Ex. 1. The l.c.m. of $x^3 - x$, $x^3 - 1$ and $x^3 + 1$; that is, of $x(x^2 - 1)$, $x^3 - 1$, and $x^3 + 1$; that is $x(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$ and $(x + 1)(x^2 - x + 1) = x(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$ $= x(x^3 - 1)(x^3 + 1) = x(x^6 - 1) = x^7 - x$.

NOTE.—Of course the same factor is only to be taken *once* in the l. c. m. although it may occur in each of the given quantities.

Ex. 2.—The l. c. m. of $4(x^3 - xy^2)$, $20(x^3 + x^2y - xy^2 - y^3)$, $12(xy^2 + y^3)$, $12(x^2 + xy)^2$ and $8(x^3 - x^2y)$;

that is, of $4x(x^2 - y^2)$; $20\{(x^3 + x^2y) - (xy^2 + y^3)\}$; $12y^2(x + y)$; $12x^2(x + y)^2$ and $8x^2(x - y)$;

that is, of $4x(x+y)(x-y)$; $20\{x^2(x+y)-y^2(x+y)\}$; $12y^2(x+y)$; $12x^2(x+y)^2$, and $8x^2(x-y)$;

that is, of $4x(x+y)(x-y)$; $20(x+y)(x^2-y^2)$; $12y^2(x+y)$; $12x^2(x+y)^2$, and $8x^2(x-y)$;

that is, of $4x(x+y)(x-y)$; $20(x+y)^2(x-y)$; $12y^2(x+y)$; $12x^2(x+y)^2$, and $8x^2(x-y)$ is equal to $120x^2y^2(x+y)^2(x-y) = 120x^2y^2(x^3+x^2y-xy^2-y^3)$

EXERCISE XXIII.

Find the l. c. m. of—

1. $2a^2x$, $3xy$, $4ab^2y$, and $-3x^2y^2$.
2. $2ax^2$, $3xy^2$, $4yz^2$, $-2a^2x$, and $-2x^2y$.
3. $(x-y)$, $(x^2-y)^2$, and $(x-y)^2$.
4. x^2-y^2, x^3-y^3, and x^4-y^4.
5. $(x-x^2)^2$, (x^2-1), and $4(1+x)x$.
6. $4(a-b)^2$, $6(a^3-b^3)$, $6(a^3+b^3)$, and $9(a^6-b^6)$.
7. (x^2-3x), $(x^2-10x+21)$, and x^2-7x.
8. (a^3-x^3), and $(a^2+x-ax-a)$.
9. $a^3-9a^2+26a-24$, and $a^3-8a^2+19a-12$.
10. $3(a^3-b^3)$, $4(a-b)^3$, $5(a^4-b^4)$, $6(a-b)^2$, and $(a^2-b^2)^3$.

SECTION V.

FRACTIONS.

94. Algebraic fractions are in all essential respects similar to arithmetical fractions, and the rules for operating upon them are the same as those for common arithmetic, and are deduced in the same manner.

95. Since the value of a fraction is the quotient, which is obtained by dividing the numerator by the denominator, we infer the following principles, upon which the principal rules are founded;—

I. *That multiplying the numerator, or dividing the denominator of a fraction by any quantity, multiplies the fraction by that quantity.*

II. *That dividing the numerator, or multiplying the denominator, of any fraction by a quantity, divides the fraction by that quantity.*

III. *That multiplying or dividing both numerator and denominator of a fraction by the same quantity does not change its value.*

96. These principles are, however, susceptible of general proof, as follows :—

I. Let $\frac{a}{b}$ be any fraction and m any integer, then $\frac{am}{b} = \frac{a}{b} \times m$. For in each of the fractions $\frac{a}{b}$ and $\frac{am}{b}$ the unit is divided into b equal parts, and m times as many of these parts are indicated by the latter fraction as by the former. Conversely $\frac{a}{b} = \frac{am}{b} \div m$.

Again, let $\frac{a}{bm}$ be any fraction and m any integer, then $\frac{a}{b} = \frac{a}{bm} \times m$. For in each of the fractions $\frac{a}{bm}$ and $\frac{a}{b}$ the same number of parts is taken; but each part of the former is $\frac{1}{m}$th of each part of the latter, therefore each part of the latter fraction is m times larger than each part of the former; and since the same number of parts is taken of each, it follows that the latter fraction $\frac{a}{b}$ is m times greater than the former fraction $\frac{a}{bm}$.

II. The proof of this is simply the converse of the above.

That is, since $\frac{am}{b} = \frac{a}{b} \times m$, conversely $\frac{a}{b} = \frac{am}{b} \div m$.

And since $\frac{a}{b} = \frac{a}{bm} \times m$, conversely $\frac{a}{bm} = \frac{a}{b} \div m$.

III. Since both multiplying and dividing any quantity by the same number does not change its value, if we both multiply and divide $\frac{a}{b}$ by m, its value will remain unaltered. But (I) $\frac{a}{b} \times m = \frac{am}{b}$, and (II) $\frac{am}{b} \div m = \frac{am}{bm} = \frac{a}{b}$, i. e., although the parts in the former fraction are each but $\frac{1}{m}$th of each of those in the latter, m times more of them are taken

Or, $\frac{am}{bm} = \frac{am^1m^{-1}}{b}$ (Art. 69, *Cor.*) $= \frac{am^0}{b} = \frac{a}{b}$ because $m^0 = 1$.

And since $a \div m = \frac{a}{m} = am^{-1}$ and $b \div m = \frac{b}{m} = bm^{-1}$. Therefore

$$\frac{a \div m}{b \div m} = \frac{am^{-1}}{bm^{-1}} = \frac{a^{am^{-1}}}{b} = \frac{am^0}{b} = \frac{a}{b} \text{ because } m^0 = 1.$$

97. The following facts should be borne in mind by the student :—

I. *Any integer may be expressed as a fraction having* 1 *for denominator.* Thus, $a = \frac{a}{1}$.

II. *Any quantity divided by itself equals unity.* Thus, $\frac{b}{b} = 1$.

III. *Any integral expresssion may be expressed as a fraction having a given denominator, the numerator being obtained by multiplying the given expression by the proposed denominator.*

Thus, let it be required to express a as a fraction with denominator b.

(Art. 97, I). $a = \frac{a}{1}$, multiply both numerator and denominator by b, we get $a = \frac{a}{1} = \frac{ab}{b}$.

IV. *The signs of all the terms of both numerator and denominator may be changed without altering the value of the expression, this being equivalent to merely multiplying both numerator and denominator by* -1.

$$\text{Thus, } \frac{2a - 3b + 4cm - x^2}{3 + 2m - y^2 - 3c} = \frac{3b - 2a - 4cm + x^2}{3c - 3 - 2m + y^2}.$$

V. *All the rules and formulæ in fractions hold whether the letters employed represent integeral or fractional, positive or negative quantities.*

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98. To reduce a fraction to its lowest terms :

RULE.—*Divide both numerator and denominator by their G.C.M.*

NOTE.—The student should always endeavour to factor the numerator and denominator so as to find by inspection the G. C. M. when it can be so found. Otherwise he must find the G. C. M. of the two terms by Art. 89.

Ex. 1. $\frac{a^2mxy}{amx^2} = \frac{amx \times ay}{amx \times x} = \frac{ay}{x}$.

Ex. 2. $\frac{a^2 + 3a^2x}{2a^2 - 3a^2m + a^2y^2} = \frac{a^2(1 + 3x)}{a^2(2 - 3m + y^2)} = \frac{1 + 3x}{2 - 3m + y^2}$.

Ex. 3. $\frac{a^4 - x^4}{a^2 - 2ax + x^2} = \frac{(a^2 + x^2)(a + x)(a - x)}{(a - x)(a - x)} = \frac{(a^2 + x^2)(a + x)}{a - x}$

$= \frac{a^3 + a^2x + ax^2 + x^3}{a - x}$.

Ex. 4. $\frac{a^2 - 6a - 27}{a^2 + 8a + 15} = \frac{(a + 3)(a - 9)}{(a + 3)(a + 5)} = \frac{a - 9}{a + 5}$.

Ex. 5. $\frac{x^2 - xy + mx - my}{x^2 + xy + mx + my} = \frac{x(x - y) + m(x - y)}{x(x + y) + m(x + y)}$

$= \frac{(x - y)(x + m)}{(x + y)(x + m)} = \frac{x - y}{x + y}$.

Ex. 6. $\frac{x^3 - 8x + 3}{x^6 + 3x^5 + x + 3}$. Here (Art. 89) the G. C. M. of the numerator and denominator is $x + 3$, and dividing both terms by $x + 3$ we get $\frac{(x^3 - 8x + 3) \div (x + 3)}{(x^6 + 3x^5 + x + 3) \div (x + 3)} = \frac{x^2 - 3x + 1}{x^5 + 1}$.

EXERCISE XXIV.

Reduce the following fractions to their lowest terms :—

1. $\frac{a^2 - ab}{ax + ay}$. 2. $\frac{2am + m^2x - m^3}{3a^2m + m^2}$. 3. $\frac{c + ac}{n + an}$.

4. $\frac{a^2b + a^2b^2 + a^2bm}{mx + bx + x}$. 5. $\frac{abc^2}{ab + bc}$. 6. $\frac{ax^2y^3}{a^2x^2m + axy + x^3y^2z^3}$.

7. $\frac{21x^2y^2 - 35x^3y^2}{14x^3y^2}$. 8. $\frac{a - m}{a^2 - m^2}$. 9. $\frac{a^3 + b^3}{a^2 - b^2}$. 10. $\frac{a^2 - 2ab + b^2}{a^3 - b^3}$.

11. $\frac{a^3 + b^3}{a^3 - b^3}$. 12. $\frac{a^6 - m^6}{(a + m)(a - m)}$. 13. $\frac{a^4 - m^4}{a^5 - a^3m^2}$.

14. $\frac{7x^2 - 21x + 35}{11x^2 - 33x + 55}$. 15. $\frac{x^2 - 11x + 28}{x^2 - 4x - 21}$. 16. $\frac{4x^2 + 12x + 9}{2x^2 - 5x - 12}$.

17. $\frac{x^3 + 2x^2y + 3x^2y^2}{2x^4 - 3x^3y - 5x^2y^2}$. 18. $\frac{a^3 - 2a^2b + 2ab^2 - b^3}{a^4 + a^2b^2 + b^4}$.

19. $\dfrac{a^4 - m^4}{a^3 - a^2m - am^2 + m^3}$. 20. $\dfrac{ac + bd + ad + bc}{am + 2bp + 2ap + bm}$.

21. $\dfrac{x^2 + (a + b)x + ab}{x^2 + (b + c)x + bc}$. 22. $\dfrac{2x^3 + x^2 - 8x + 5}{7x^2 - 12x + 5}$.

23. $\dfrac{(a + m)(a + m + x)(a + m - x)}{2a^2m^2 + 2a^2x^2 + 2m^2x^2 - a^4 - m^4 - x^4}$. 24. $\dfrac{a^{12} + x^{12}}{a^{20} + x^{20}}$.

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99. To reduce a mixed quantity to a fractional form :—

Rule.—*Multiply the entire part of the quantity by the denominator of the fraction, and to the product connect the numerator of the fractional part by its proper sign. Beneath the whole expression thus formed, write the denominator.*

Ex. 1. $a - b + \dfrac{x + y}{am} = \dfrac{a^2m - abm + (x + y)}{am} = \dfrac{a^2m - abm + x + y}{am}$.

Ex. 2. $a^2 - 2ay - \dfrac{3x - 2am}{4y^2} = \dfrac{4a^2y^2 - 8ay^3 - (3x - 2am)}{4y^2}$

$= \dfrac{4a^2y^2 - 8ay^3 - 3x + 2am}{4y^2}$.

Exercise XXV.

Reduce the following mixed quantities to their equivalent fractions :—

1. $2ax - y + \dfrac{3 - 2a}{ax}$. 2. $a^2 + a + 1 + \dfrac{2}{a - 1}$. 3. $3a - y - \dfrac{3a^2 - 30}{x + 3}$.

4. $3a + y - \dfrac{2a + xy}{x - y}$. 5. $3ax - y^2 + m - \dfrac{3ax^2 + xy^2}{a + x}$.

6. $xy + mz + \dfrac{xyz - z^2m - 2m^2z}{z + 2m}$. 7. $(a + b)^2 - \dfrac{(a - b)^3}{a + b}$.

8. $1 - \dfrac{a^2 - m^2}{a^2 + m^2}$. 9. $1 - \dfrac{a^2 - 2ax + x^2}{a^2 + x^2}$.

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100. To reduce a fraction to a mixed quantity :—

Rule.—*Divide the numerator by the denominator, and place the remainder, if any, over the denominator for the fractional part. Connect the fraction thus obtained to the entire part of the quotient by the sign plus.*

Ex. 1. $\dfrac{3a^2 - 12ab + y - 9a}{3a} = a - 4b - 3 + \dfrac{y}{3a}$.

Ex. 2. $\dfrac{6x^2 - ax}{3x + 1} = 2x + \dfrac{-2x - ax}{3x + 1} = 2x - \dfrac{2x + ax}{3x + 1}$.

EXERCISE XXVI.

Reduce the following fractions to mixed quantities:—

1. $\dfrac{20m^2 - 20m + 1}{5m}$. 2. $\dfrac{a^2 + x^2}{a - x}$. 3. $\dfrac{x^2 + 2xy + y^2 + x^3 - y^4}{x + y}$.

4. $\dfrac{5m^3 - 5p^3 + 3}{m - p}$. 5. $\dfrac{1 - a - ab + a^2b}{ab - b}$. 6. $\dfrac{m + ab + 5am}{m + b}$.

101. To reduce fractions to a common denominator:—

RULE.—*Find the l. c. m. of all the denominators; then taking each fraction in succession, divide this l. c. m. by the denominator, and multiply both terms by the quotient thus obtained.*

Ex. 1. Reduce $\dfrac{1}{a}$, $\dfrac{b}{m}$, and $\dfrac{c}{mx}$ to a com. denom.

The l. c. m. of a, m, and $mx = amx$.

$amx \div a = mx$ = multiplier for both terms of 1st fraction,
$amx \div m = ax$ = multiplier for both terms of 2nd fraction,
$amx \div mx = a$ = multiplier for both terms of 3rd fraction,

$\dfrac{1 \times mx}{a \times mx} = \dfrac{mx}{amx}$. $\dfrac{b \times ax}{m \times ax} = \dfrac{abx}{amx}$. $\dfrac{c \times a}{mx \times a} = \dfrac{ac}{amx}$.

Hence the required fractions are $\dfrac{mx}{amx}$, $\dfrac{abx}{amx}$, and $\dfrac{ac}{amx}$.

Ex. 2. Reduce $\dfrac{1 + a}{1 - a}$, $\dfrac{1 + a^2}{1 - a^2}$, and $\dfrac{1 + a^3}{1 - a^3}$ to equivalent fractions having a common denominator.

OPERATION.

The l. c. m. of $1 - a$, $1 - a^2$, and $1 - a^3 = (1 + a)(1 - a^3) = (1 + a)(1 - a)(1 + a + a^2)$.

1st multiplier = l. c. m. $\div (1 - a) = (1 + a)(1 + a + a^2)$;
2nd " = " $\div (1 - a^2) = 1 + a + a^2$; and
3rd " = " $\div (1 - a^3) = (1 + a)$.

Using these multipliers the three given fractions become

$$\frac{(1 + a)(1 + a)(1 + a + a^2)}{(1 - a)(1 + a)(1 + a + a^2)};\ \frac{(1 + a^2)(1 + a + a^2)}{(1 - a^2)(1 + a + a^2)},\ \text{and}\ \frac{(1 + a^3)(1 + a)}{(1 - a^3)(1 + a)}$$

$$= \frac{(1 + a)^2(1 + a + a^2)}{1 + a - a^3 - a^4};\ \frac{(1 + a^2)(1 + a + a^2)}{1 + a - a^3 - a^4},\ \text{and}\ \frac{(1 + a^3)(1 - a^2)}{1 + a - a^3 - a^4}.$$

EXERCISE XXVII.

Reduce the following fractions to others having a common denominator :—

1. $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}$, and $\frac{x}{m}$. 2. $\frac{1}{m}, \frac{a}{xy}$, and $\frac{b}{mx}$. 3. $\frac{2}{3a}, \frac{a}{4b}$, and $\frac{m}{2xy}$.

4. $\frac{1+m}{1-m}$ and $\frac{1-m}{1+m}$. 5. $\frac{x^2-y^2}{x^2+y^2}$ and $\frac{x+y}{x^3+xy^2}$.

6. $\frac{3x}{x-y}, \frac{4x+y}{x^2-y^2}$, and $\frac{2x-3y}{2(x+y)}$. 7. $\frac{3a}{2+x}, \frac{4-2x}{3m}$, and $\frac{1}{2a^2}$.

8. $a, \left(\frac{4x}{3}\right), \left(\frac{x^2+1}{x^2-1}\right)$, and $\left(x+\frac{2}{3}\right)$.

9. $\frac{1}{a(a+b)}, \frac{1}{3a^2(a^2-b^2)}$, and $\frac{1}{6a^3(a+b)}$.

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102. To add or subtract algebraic fractions :—

RULE.—*Reduce them to a common denominator, then add or subtract the numerators, and beneath the sum or difference place the common denominator.*

Ex. 1. $$\frac{1-a}{1+a}+\frac{1}{1-a}+\frac{a^2}{1-a^2}=\frac{(1-a)^2}{1-a^2}+\frac{1+a}{1-a^2}+\frac{a^2}{1-a^2}$$
$$=\frac{1-2a+a^2+1+a+a^2}{1-a^2}=\frac{2-a+2a^2}{1-a^2}.$$

Ex. 2. $$\frac{1+x^2}{1-x^2}-\frac{1-x^2}{1+x^2}=\frac{(1+x^2)^2}{1-x^4}-\frac{(1-x^2)^2}{1-x^4}=\frac{1+2x^2+x^4}{1-x^4}$$
$$-\frac{1-2x^2+x^4}{1-x^4}=\frac{1+2x+x^4-1+2x-x^4}{1-x^4}=\frac{4x}{1-x^4}.$$

Ex. 3. $$\frac{a}{1-a}-\frac{a^2}{(1-a)^2}+\frac{a^3}{(1-a)^3}=\frac{a(1-a)^2}{(1-a)^3}-\frac{a^2(1-a)}{(1-a)^3}$$
$$+\frac{a^3}{(1-a)^3}=\frac{a(1-a)^2-a^2(1-a)+a^3}{(1-a)^3}=\frac{a(1-2a+a^2)-(a^2-a^3)+a^3}{(1-a)^3}$$
$$=\frac{a-2a^2+a^3-a^2+a^3+a^3}{(1-a)^3}=\frac{a-3a^2+3a^3}{(1-a)^3}.$$

Ex. 4. $\frac{(x^2+y^2)^2}{xy(x-y)^2} - \frac{y}{x} - 2 - \frac{x}{y} = \frac{x^4+2x^2y^2+y^4}{xy(x-y)^2} - \frac{y^2(x-y)^2}{xy(x-y)^2}$

$- \frac{2xy(x-y)^2}{xy(x-y)^2} - \frac{x^2(x-y)^2}{xy(x-y)^2}$

$= \frac{x^4+2x^2y^2+y^4-y^2(x^2-2xy+y^2)-2xy(x^2-2xy+y^2)-x^2(x^2-2xy+y^2)}{xy(x-y)^2}$

$= \frac{x^4+2x^2y^2+y^4-x^2y^2+2xy^3-y^4-2x^3y+4x^2y^2-2xy^3-x^4+2x^3y-x^2y^2}{xy(x-y)^2}$

$= \frac{4x^2y^2}{xy(x-y)^2} = \frac{4xy}{(x-y)^2}.$

EXERCISE XXVIII.

Find the value of:—

1. $\frac{2a}{b} + \frac{3}{2b} - \frac{c}{m}$. 2. $\frac{x}{y} + \frac{2(a-b)}{y^2(x+3)}$. 3. $\frac{a-b}{a+b} - \frac{a+b}{a-b}$.

4. $5x - \frac{2x}{7} + \frac{5x}{9} + x^2$. 5. $\frac{x^3}{(x+y)^3} + \frac{y}{x+y} - \frac{xy}{(x+y)^2}$.

6. $\frac{a-b}{ab} + \frac{b-c}{bc} - \frac{a-c}{ac}$. 7. $\frac{m}{m+p} - \frac{p}{m-p}$.

8. $\frac{3}{1+2a} - \frac{4(1-5a)}{4a^2-1} - \frac{7}{2a-1}$. 9. $\frac{x(16-x)}{x^2-4} + \frac{2x+3}{2-x} - \frac{2-3x}{x+2}$.

10. $\frac{1}{a}(x+y) + \frac{1}{b}(x+y) - \left(\frac{x+y}{a} - \frac{x-y}{b}\right)$.

11. $\frac{m+p}{(p-x)(x-m)} + \frac{p+x}{(x-m)(m-p)} + \frac{m+x}{(m-p)(p-x)}$.

12. $\frac{a-b}{a+b} + \frac{b-c}{b+c} - \frac{2ab-2ac}{b(a+c)+c(a+b)-b(c-b)}$.

13. $\frac{1}{1-x} - \frac{1}{1+x} + \frac{3}{1-2x} - \frac{3}{1+2x}$.

14. $\frac{m}{a(a-b)(a-c)} + \frac{m}{b(b-a)(b-c)} + \frac{m}{c(c-a)(c-b)}$.

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103. To multiply fractions together:—

RULE.—*Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.*

NOTE 1.—If any of the given quantities are mixed fractions, they must be reduced to the fractional form before multiplying.

NOTE 2.—Before multiplying the student must, by attention to the principles given in (Arts. 70, 80,) strike out all the factors common to a numerator and a denominator.

PROOF OF RULE.—Let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$

Let $\frac{a}{b} = x$ and $\frac{c}{d} = y$, then $\frac{a}{b} \times \frac{c}{d} = xy$. Also $a = bx$ and $c = dy$.

Hence $ac = bdxy$, and dividing each of these by bd we get $\frac{ac}{bd} = xy$

But $\frac{a}{b} \times \frac{c}{d} = xy$ Therefore $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = \frac{\textit{product of numerators.}}{\textit{product of denominators}}$

Ex. 1. $\frac{1-a}{x+y} \times \frac{a}{b} = \frac{(1-a)a}{(x+y)b} = \frac{a-a^2}{bx+by}$.

Ex. 2. $\frac{x^5 - b^2x^3}{x^3+b^3} \times \frac{x^2+bx}{x-b} \times \frac{x^2-bx+b^2}{x^2}$

$$= \frac{x^3(x^2-b^2) \times x(x+b) \times (x^2-bx+b^2)}{(x^3+b^3)(x-b)x^2}$$

$$= \frac{x^3(x-b)(x+b) \times x(x+b) \times (x^2-bx+b^2)}{(x+b)(x^2-bx+b^2) \times (x-b) \times x^2}$$

$$= \frac{x^2(x+b)}{1} = x^3 + bx^2.$$

Ex. 3. $\left(a - \frac{1}{a}\right) \times \left(1 - \frac{1}{a}\right)^2 = \frac{a^2-1}{a} \times \left(\frac{a-1}{a}\right)^2 = \frac{a^2-1}{a}$

$$\times \frac{a-1}{a} \times \frac{a-1}{a} = \frac{(a-1)^2(a^2-1)}{a^3} = \frac{a^4-2a^3+2a-1}{a^3}.$$

EXERCISE XXIX.

Find the value of :—

1. $\frac{2x}{5} \times \frac{3x}{2a}$. 2. $\frac{2m}{xy} \times \frac{x^2}{my} \times \frac{y^2}{x}$. 3. $\frac{2(a+b)}{xy} \times \frac{x(a-b)}{3b+3a}$.

4. $3a \times \frac{x+1}{2a} \times \frac{x-1}{a+b}$. 5. $\frac{a^2-x^2}{a+b} \times \frac{a^2-b^2}{a+x} \times \frac{a}{x(a-x)}$.

6. $\frac{a^2-m^2}{my} \times \frac{a^2+m^2}{m-a}$ 7. $\frac{a^2-x^2}{3ax} \times \frac{4ax^2}{a+x}$.

8. $\dfrac{x^2-13x+42}{x^2-5x} \times \dfrac{x^2-9x+20}{x^2-6x}$. 9. $\dfrac{a}{by} \times \dfrac{b}{cy^2} \times \dfrac{c}{dy^3} \times \dfrac{d}{fy^4} \times \dfrac{n}{fy^5}$.

10. $\dfrac{a^2-4}{a^2-1} \times \dfrac{a^2-1}{2a} \times \dfrac{a-2}{2+a}$.

11. $\dfrac{x^2-a^2}{x^2+bx-ax-ab} \times \dfrac{x^2+bx+cx+bc}{x^2+cx+dx+cd}$.

12. $\dfrac{x^2+x-12}{x^2-13x+40} \times \dfrac{x^2+2x-35}{x^2-7x-44}$. 13. $(1-a+a^2) \times (1+\dfrac{1}{a}+\dfrac{1}{a^2})$.

14. $\dfrac{4a^2-16m^2}{a-2m} \times \dfrac{5a}{20a^2+80am+80m^2} \times \dfrac{a+2m}{a}$.

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104. To divide one algebraic fraction by another :—

RULE.—*Invert the divisor and proceed as in multiplication.*

NOTE 1.—If either of the given quantities be a mixed fraction it must be reduced to a fractional form before applying the rule.

NOTE 2.—After having inverted the terms of the divisor, be careful to cancel as far as possible before multiplying.

PROOF OF RULE FOR DIVISION.—Let it be required to divide $\dfrac{a}{b}$ by $\dfrac{c}{d}$.

Put $\dfrac{a}{b} \div \dfrac{c}{d} = x$; multiplying each of these by $\dfrac{c}{d}$ we get $\dfrac{a}{b} = x \times \dfrac{c}{d}$ $= \dfrac{cx}{d}$. Again multiplying each of these by d we get $\dfrac{ad}{b} = cx$, therefore $x = \dfrac{ad}{bc}$. But $x = \dfrac{a}{b} \div \dfrac{c}{d}$; therefore $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{ad}{bc} = \dfrac{a}{b} \times \dfrac{d}{c}$ = dividend $\times$ divisor with terms inverted.

Ex. 1. $\dfrac{a^2-b^2}{a^2+b^2} \div \dfrac{a^2-2ab+b^2}{a^4-b^4} = \dfrac{a^2-b^2}{a^2+b^2} \times \dfrac{a^4-b^4}{a^2-2ab+b^2}$

$$= \frac{(a-b)(a+b) \times (a^2+b^2)(a-b)(a+b)}{(a^2+b^2)(a-b)(a-b)} = (a+b)^2.$$

Ex. 2. $\dfrac{a^3+y^3}{a^2-y^2} \div \dfrac{a^2-ay+y^2}{(a-y)^2} = \dfrac{a^3+y^3}{a^2-y^2} \times \dfrac{(a-y)^2}{a^2-ay+y^2}$

$$= \frac{(a+y)(a^2-ay+y^2)(a-y)(a-y)}{(a-y)(a+y)(a^2-ay+y^2)} = a-y.$$

EXERCISE XXX.

Find the value of:—

1. $\dfrac{1}{x} \div \dfrac{x}{y}$. 2. $\dfrac{a+x}{a} \div \left(1 - \dfrac{x}{a}\right)$. 3. $\dfrac{a+b}{a-b} \div \dfrac{a^2+2ab+b^2}{a^2-2ab+b^2}$.

4. $\left(\dfrac{a^4-x^4}{2+y} \times \dfrac{y^2-4}{a-x}\right) \div \dfrac{a^2+x^2}{3a}$. 5. $\dfrac{x-3}{x-9} \div \dfrac{x^2-15x+56}{x^2-17x+72}$.

6. $\left(\dfrac{a}{a+b} + \dfrac{b}{a-b}\right) \div \left(\dfrac{a}{a-b} - \dfrac{b}{a+b}\right)$.

7. $\left(\dfrac{a^6-x^6}{a^2-2ax+x^2} \times \dfrac{1}{a+x}\right) \div \left(\dfrac{a^2+ax+x^2}{a-x} \times \dfrac{a^2-ax+x^2}{1}\right)$.

8. $\dfrac{3a^2-3}{2(a+b)} \div \dfrac{x^2-1}{2a^2+2ab}$. 9. $\left(1 + \dfrac{y}{x+y} + \dfrac{x}{y}\right) \div \left(2 + \dfrac{x}{y} - \dfrac{x}{x+y}\right)$.

10. $\left(\dfrac{a^2+b^2}{a^2-b^2} - \dfrac{a^2-b^2}{a^2+b^2}\right) \div \left(\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b}\right)$.

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105. To reduce complex algebraic fractions to simple fractions :—

RULE.—*Reduce both numerator and denominator to simple fractions, if they be not simple already; then having thus reduced the whole expression to the form of* $\dfrac{\textit{fraction}}{\textit{fraction}}$, *multiply the extremes together for a numerator, and the means together for a denominator.*

Ex. 1. $$\frac{1-\frac{1}{3}x}{\frac{1}{4}y-a} = \frac{\dfrac{3-x}{3}}{\dfrac{y-4a}{4}} = \frac{4(3-x)}{3(y-4a)} = \frac{12-4x}{3y-12a}.$$

Ex. 2. $$\frac{\dfrac{1}{1+a}}{1+\dfrac{1}{1-a}} = \frac{\dfrac{1}{1+a}}{\dfrac{2-a}{1-a}} = \frac{1-a}{(1+a)(2-a)} = \frac{1-a}{2+a-a^2}$$

Ex. 3. $$\frac{\dfrac{a}{1+\dfrac{1}{a-1}}}{1+\dfrac{a}{a^2-\dfrac{1}{1-\dfrac{a-1}{a}}}} = \frac{\dfrac{a}{\dfrac{a}{a-1}}}{1+\dfrac{a}{a^2-\dfrac{1}{\dfrac{1}{a}}}} = \frac{\dfrac{a-1}{1}}{1+\dfrac{a}{a^2-a}}$$

$$= \frac{\dfrac{a-1}{1}}{\dfrac{a^2}{a^2-a}} = \frac{(a-1)^2}{a} = \frac{a^2-2a+1}{a}.$$

EXERCISE XXXI.

Simplify the following complex fractions :—

1. $\dfrac{\frac{1}{3}(a-b)}{\frac{2}{3}a+\frac{3}{5}b}$. 2. $\dfrac{a-\frac{2}{7}x}{3}$. 3. $\dfrac{x}{1+\dfrac{2x}{a}}$. 4. $\dfrac{2\frac{1}{4}-\frac{3}{8}(x+2)}{1\frac{1}{3}+\frac{1}{2}(x-3)}$.

5. $\dfrac{1\frac{1}{2}-\frac{3}{5}(x-a)}{\frac{2}{3}(a+x)-\frac{2}{5}}$. 6. $\dfrac{\dfrac{1+2a}{1-2a}-\dfrac{1-2a}{1+2a}}{\dfrac{1-2a}{1+2a}+\dfrac{1+2a}{1-2a}}$. 7. $\dfrac{\dfrac{1}{1+a}-\dfrac{1}{1-a}}{\dfrac{1}{1-a}+\dfrac{1}{1+a}}$.

8. $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{1}{b}-\dfrac{1}{a}} \div \dfrac{a^3+b^3}{a^2-b^2}$. 9. $\dfrac{xy-\dfrac{1+x^2y^2}{xy}}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{xy}}}}$.

10. $\dfrac{\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}}{\dfrac{adf-ac}{bdf+be+cf}}$. 11. $\dfrac{\dfrac{(1-2m)^2+(2m+1)^2}{(1-4m^2)-(1-2m)^2}}{\dfrac{(1+2m)^2-(1-4m^2)}{(1-2m)^2-(2m+1)^2}}$.

106. THEOREM.—*If any two fractions are equal to one another, we may combine, in any manner whatever, by addition and subtraction, the numerator and denominator of the one, provided we at the same time similarly combine the numerator and denominator of the other, and the resulting fractions will be equal.*

That is, if $\frac{a}{b} = \frac{c}{d}$, then

$$\frac{a+b}{b} = \frac{c+d}{d} \text{ (I)}; \quad \frac{a-b}{b} = \frac{c-d}{d} \text{ (II)}; \quad \frac{b}{a} = \frac{d}{c} \text{ (III)},$$

$$\frac{a}{c} = \frac{b}{d} \text{ (IV)}; \quad \frac{a+b}{a} = \frac{c+d}{c} \text{ (V)}; \quad \frac{a-b}{a} = \frac{c-d}{c} \text{ (VI)};$$

$$\frac{a+b}{a-b} = \frac{c+d}{c-d} \text{ (VII)}; \quad \frac{a}{a+b} = \frac{c}{c+d} \text{ (VIII)}; \quad \frac{a}{a-b} = \frac{c}{c-d} \text{ (IX)};$$

$$\frac{a}{c} = \frac{a \pm b}{c \pm d} \text{ (X)}; \quad \frac{a+b}{c+d} = \frac{a-b}{c-d} \text{ (XI)}, \text{ \&c., \&c.}$$

Also, $\frac{ma}{nb} = \frac{mc}{nd}$ (XII); $\frac{ma \pm nb}{nb} = \frac{mc \pm nd}{nd}$ (XIII);

$$\frac{ma \pm nb}{b} = \frac{mc \pm nd}{d} \text{ (XIV)}; \quad \frac{ma \pm nb}{pa \pm qb} = \frac{mc \pm nd}{pc \pm qd} \text{ (XV)}, \text{ \&c.}$$

Or, *The above propositions hold with any multiples whatever of the two numerators, and also any multiples whatever of the two denominators.*

Also, $\frac{a^n}{b^n} = \frac{c^n}{d^n}$ (XVI). *That is, the above theorem is true of any similar combinations of the same powers of the numerator and denominator.*

DEMONSTRATION.

(I). Since $\frac{a}{b} = \frac{c}{d}$ $\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$ or $\frac{a+b}{b} = \frac{c+d}{d}$.

(II). Since $\frac{a}{b} = \frac{c}{d}$ $\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$ or $\frac{a-b}{b} = \frac{c-d}{d}$.

(III). Since $\frac{a}{b} = \frac{c}{d}$ $\therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d}$ or $1 \times \frac{b}{a} = 1 \times \frac{d}{c}$

that is, $\frac{b}{a} = \frac{d}{c}$.

(IV). Since $\frac{a}{b} = \frac{c}{d}$ $\therefore \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$ or $\frac{a}{c} = \frac{b}{d}$.

(V). Since (I) $\frac{a+b}{b} = \frac{c+d}{d}$ and (III) $\frac{b}{a} = \frac{d}{c}$ $\therefore \frac{a+b}{b} \times \frac{b}{a} = \frac{c+d}{d} \times \frac{d}{c}$ or $\frac{a+b}{a} = \frac{c+d}{c}$.

(VI). Since (II) $\frac{a-b}{b} = \frac{c-d}{d}$ and (III) $\frac{b}{a} = \frac{d}{c}$ $\therefore \frac{a-b}{b} \times \frac{b}{a} = \frac{c-d}{d} \times \frac{d}{c}$ or $\frac{a-b}{a} = \frac{c-d}{c}$.

(VII). Since (II) $\frac{a-b}{b} = \frac{c-d}{d}$ $\therefore$ inverting by (III) $\frac{b}{a-b} = \frac{d}{c-d}$ and also (I) $\frac{a+b}{b} = \frac{c+d}{d}$ $\therefore$ $\frac{a+b}{b} \times \frac{b}{a-b} = \frac{c+d}{d} \times \frac{d}{c-d}$ or $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

(VIII). Since (V) $\frac{a+b}{a} = \frac{c+d}{c}$ $\therefore$ (III) $\frac{a}{a+b} = \frac{c}{c+d}$.

(IX). Since (VI) $\frac{a-b}{a} = \frac{c-d}{c}$ $\therefore$ (III) $\frac{a}{a-b} = \frac{c}{c-d}$.

(X). Since (VIII) and (IX) $\frac{a}{a \pm b} = \frac{c}{c \pm d}$ $\therefore$ alternately by (IV) $\frac{a}{c} = \frac{a \pm b}{c \pm d}$.

(XI). Since (X) $\frac{a+b}{c+d} = \frac{a}{c}$ and also $\frac{a-b}{c-d} = \frac{a}{c}$ $\therefore$ (AX. XI) $\frac{a+b}{c+d} = \frac{a-b}{c-d}$ or XI = VII taken alternately.

(XII). Since $\frac{a}{b} = \frac{c}{d}$ $\therefore \frac{a}{b} \times \frac{m}{n} = \frac{c}{d} \times \frac{m}{n}$ or $\frac{ma}{nb} = \frac{mc}{nd}$.

(XIII, &c.) Since $\frac{ma}{nb} = \frac{mc}{nd}$ $\therefore$ all the above changes may be made on these fractions.

(XVI). Since $\frac{a}{b} = \frac{c}{d}$ $\therefore \frac{a}{b} \times \frac{a}{b} = \frac{c}{d} \times \frac{c}{d}$ or $\frac{a^2}{b^2} = \frac{c^2}{d^2}$.

Similarly $\frac{a^3}{b^3} = \frac{c^3}{d^3}$ and $\frac{a^n}{b^n} = \frac{c^n}{d^n}$. And all the above changes may be made on the equal fractions $\frac{a^n}{b^n} = \frac{c^n}{d^n}$.

107. THEOREM.—*If there be any number of equal fractions, then we may combine in any manner whatever by addition or subtraction the numerators, or any multiples of the numerators, provided we similarly combine the denominators, or the same multiples of the denominators, and the resulting fractions will be equal to any one of the given fractions and to one another.*

That is, if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$,

Then $\frac{a}{b} = \frac{a \pm c \pm e}{b \pm d \pm f} = \frac{ma \pm nc \pm pe}{mb \pm nd \pm pf}$.

DEMONSTRATION. Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = x$. $\therefore a = bx$, $c = dx$, and $e = fx$, $\therefore a \pm c \pm e = bx \pm dx \pm fx = (b \pm d \pm f)x$.

Dividing each of these equals by $(b \pm d \pm f)$ we get $x = \frac{a \pm c \pm e}{b \pm d \pm f}$, but $x = \frac{a}{b}$ $\therefore$ $\frac{a}{b} = \frac{a \pm c \pm e}{b \pm d \pm f}$.

Again, since $a = bx$, $c = dx$, and $e = fx$,

$\therefore$ $ma = mbx$, $nc = ndx$, and $pe = pfx$.

And $ma \pm nc \pm pe = mbx \pm ndx \pm pfx = (mb \pm nd \pm pf)$

$\therefore$ $x = \frac{ma \pm nc \pm pe}{mb \pm nd \pm pf}$ but $x = \frac{a}{b}$ $\therefore$ $\frac{a}{b} = \frac{ma \pm nc \pm pe}{mb \pm nd \pm pf}$.

It follows that if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then $\frac{a^n}{b^n} = \frac{c^n}{d^n} = \frac{e^n}{f^n}$, and therefore $\frac{a^n}{b^n} = \frac{a^n \pm c^n \pm e^n}{b^n \pm d^n \pm f^n}$, and therefore also $\frac{a^n}{b^n} = \frac{ma^n \pm nc^n \pm pe^n}{mb^n \pm nd^n \pm pf^n}$.

SECTION VI.

SIMPLE EQUATIONS.

108. An *equation* consists of two algebraic expressions connected by the sign of equality.

Thus, $3a + x = b - m^2$; $x^3 - x^2 + 3 = \pm \sqrt{ab - m}$; $ax - b = 0$ are equations.

NOTE.—The part that precedes the sign of equality is called *the first member* or *left hand side* of the equation; the part that follows the sign of equality is called *the second member*, or *right hand side* of the equation.

109. An *identical equation*, or an *identity* as it is termed, is an equation such that *any values whatever* may be substituted for the letters it involves without destroying the equality of the two members.

Thus,
$$\left.\begin{aligned} a^2 - x^2 &= (a - x)(a + x) \\ 4(a + b)(a + b) &= 4a^2 + 8ab + 4b^2 \\ \tfrac{1}{2}(a + x) + \tfrac{1}{2}(a - x) &= a \end{aligned}\right\}$$
are identities, because no matter what numerical value may be assigned to a and x or to a and b, the members are equal to one another.

110. All other equations are called *equations of condition*, and the equality existing between the members holds only when particular values are assigned to some particular letter or letters involved.

111. The letter or letters for which such particular values must be found, are called the *unknown quantities*, and are generally represented by the last letters of the alphabet, x, y, z, w, &c.

112. An equation is said to be *satisfied* by any value which may be substituted for the unknown quantity without destroying the equality of the two members of the equation.

113. The *solution* of the equation is the process of finding such values for the unknown quantity or quantities as shall satisfy the equation.

114. A *root* of an equation is any value of the unknown quantity by which the equation is satisfied.

Thus, 4 is the root of the equation $x - 3 = 1$.
$1\frac{1}{5}$ and $-\frac{2}{5}$ are the roots of the equation $25x^2 - 20x - 12 = 0$.
2, 5, and -7, are the roots of the equation $x^3 - 39x = -70$.

115. An equation which involves only one unknown quantity is said to be of as many dimensions as is indicated by the exponent of the highest power of the unknown quantity that occurs in it.

$$\left.\begin{array}{l} \text{Thus, } 4x - 3 = 11 \\ 2a(x - m) + x = b^2 - m \end{array}\right\} \text{are equations \textit{of one dimension} or \textit{simple} equations, or equations of the \textit{first} degree.}$$

$$\left.\begin{array}{l} 6x^2 - x + 30 = 0 \\ cx^2 + 2ax = b \end{array}\right\} \text{are equations of \textit{two dimensions}, or \textit{quadratic} equations, or equations of the \textit{second degree}.}$$

$$\left.\begin{array}{r} 4x^3 - 112x^2 + 109x - 27 = 0 \\ x^3 - 15x^2 + 74x - 120 = 0 \end{array}\right\} \text{are equations of \textit{three dimensions}, or \textit{cubic} equations, or equations of the \textit{third degree}.}$$

$$\left.\begin{array}{l} x^4 - 74x^2 + 1225 = 0 \\ x^4 - 4x^3 + 6x^2 - 4x - 5 = 0 \end{array}\right\} \text{are equations of \textit{four dimensions}, or \textit{biquadratic} equations, or equations of the \textit{fourth degree}.}$$

116. It will be shown hereafter that an equation involving only one unknown, has as many roots as it has dimensions, and *only* as many.

Thus, a simple equation has only one root.
a quadratic equation has only two roots.
a cubic equation has only three roots, &c.

117. The solution of simple equations involves the following principles:—

I. *Any term may be carried from one side of the equation to the other, or* TRANSPOSED, *as it is termed, by changing its sign.*

Thus, if $4x - a = 7 + m$, then $4x = 7 + m + a$, this being equivalent to adding $+ a$ to each side of the equation (Ax. II).

So if $2x - a = 4b + x$, then $2x - x = 4b + a$, this being equivalent to adding $+ a$ and $- x$ to each side (Ax. II)

II. *The signs of all the terms of both members may be changed without altering the equality of the two members.*

Thus, if $3a - 4x + b = -m + ax - c$,
Then also, $-3a + 4x - b = m - ax + c$.

NOTE.—This is equivalent to transposing every term, or to multiplying both sides of the equation by -1, which of course does not affect the equality.

III. *An equation, any of whose terms involve fractions, may be cleared of these fractions, i. e., converted into another equation not involving fractions, by multiplying each member by the l. c. m. of all the denominators of the fractions.*

Thus, if $\frac{x}{2} + \frac{x}{3} + \frac{x}{5} + \frac{x}{6} = 20$, and we multiply each side by 30, which is the l. c. m. of the denominators, we get $15x + 10x + 6x + 5x = 600$.

NOTE.—This is merely multiplying both members of the equation by the *same quantity*, and, of course (Ax: IV), does not destroy the equality.

IV. *Both members of an equation may be divided by the same quantity without destroying the equality. Hence, having reduced an equation by the foregoing principles, should the unknown quantity have a coefficient, we may divide each member by that coefficient.*

Thus, if $11x = 44$, then dividing each member by 11 we get $x = 4$

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118. THEOREM.—*A simple equation, or equation of the first degree, involving only one unknown, can have only one root.*

DEMONSTRATION.—By transposing all the known quantities to the right hand member, and the unknown quantities to the left hand member, every simple equation can be reduced to the form $ax = b$.

If it be possible let $ax = b$, have two dissimilar roots β and γ.

Then $a\beta = b$ and also $a\gamma = b$, and by subtraction $a\beta - a\gamma = b - b = 0$, that is, $a(\beta - \gamma) = 0$, which is absurd, because, by supposition, $\beta - \gamma$ does not $= 0$, nor does $a = 0$.

Therefore $ax = b$ cannot have two roots.

119. From Art. 117 we get the following rule for solving a simple equation involving only one unknown quantity.

I. *Clear the equation of fractions, and, if necessary, of brackets also.*

II. *Transpose all the terms involving the unknown quantity to the left hand member of the equation, and the remaining terms to the right hand member.*

III. *Collect, by addition and subtraction, as far as possible, the several terms of each member into one term, i.e., reduce each member to its simplest form.*

IV. *Divide each member by the coefficient of the unknown quantity.*

Ex. 1. Given $8x + 7 = 2x + 43$, to find the value of x.

SOLUTION.

$8x + 7 = 2x + 43$	(I)	
$8x - 2x = 43 - 7$	(II)	= (I) transposed.
$6x = 36$	(III)	= (II) collected.
$x = 6$	(IV)	= (III) $\div$ 6.

Ex. 2. Given $\frac{x}{2} + \frac{x}{3} = \frac{x}{4} + 14$ to find the value of x.

SOLUTION.

$\frac{x}{2} + \frac{x}{3} = \frac{x}{4} + 14$	(I)	
$6x + 4x = 3x + 168$	(II)	= (I) $\times$ 12, the l.c.m. of 2, 3, 4.
$6x + 4x - 3x = 168$	(III)	= (II) transposed.
$7x = 168$	(IV)	= (III) collected.
$x = 24$	(V)	= (IV) $\div$ 7.

Ex. 3. Given $3x + \frac{2x + 6}{5} = 5 + \frac{11x - 37}{2}$ to find the value of x.

SOLUTION.

$3x + \frac{2x + 6}{5} = 5 + \frac{11x - 37}{2}$	(I)	
$30x + 4x + 12 = 50 + 55x - 185$	(II)	= (I) $\times$ 10 (l.c.m. of 2 and 5).
$30x + 4x - 55x = 50 - 185 - 12$	(III)	= (II) transposed.
$-21x = -147$	(IV)	= (III) collected.
$x = 7$	(V)	= (IV) $\div -21$.

Ex. 4. Given $x + \frac{27 - 9x}{4} - \frac{5x + 2}{6} = 5\frac{1}{12} - \frac{2x + 5}{3} - \frac{29 + 4x}{12}$ to find the value of x.

SOLUTION.

$$x + \frac{27 - 9x}{4} - \frac{5x + 2}{6} = 5\tfrac{1}{12} - \frac{2x + 5}{3} - \frac{29 + 4x}{12} \quad (\text{I}).$$

$12x + 81 - 27x - 10x - 4 = 61 - 8x - 20 - 29 - 4x$	(II)*	= (I) × 12.
$12x - 27x - 10x + 8x + 4x = 61 - 20 - 29 + 4 - 81$	(III)	= (II) transposed.
$-13x = -65$	(IV)	= (III) collected.
$x = 5$	(V)	= (IV) ÷ − 13.

* NOTE.—The student must remember that the separating line of a fraction acts as a vinculum to the numerator, and that in clearing of fractions a minus sign before the fraction has the effect of changing all the signs of the numerator.

Ex. 5. Given $\frac{6x + 1}{15} - \frac{2x - 1}{5} = \frac{2x - 4}{7x - 16}$.

SOLUTION.

$\frac{6x + 1}{15} - \frac{2x - 1}{5} = \frac{2x - 4}{7x - 16}$	(I)	
$6x + 1 - 6x + 3 = \frac{30x - 60}{7x - 16}$	(II)*	= (I) × 15.
$4 = \frac{30x - 60}{7x - 16}$	(III)	= (II) collected.
$28x - 64 = 30x - 60$	(IV)	= (III) × $(7x - 16)$
$28x - 30x = -60 + 64$	(V)	= (IV) transposed.
$-2x = 4$	(VI)	= (V) collected.
$x = -2$	(VII)	= (VI) ÷ − 2.

* NOTE.—When one of the denominators is a binomial or trinomial, it is commonly best to first multiply each member by the l. c. m. of the other denominators, and reduce the resulting equation as much as possible before multiplying by this compound denominator. This is especially the case when one of the remaining denominators contains the others, as in this example.

Ex. 6. Given $\frac{1}{4}(x - \frac{n}{6}) = \frac{1}{3}(x - \frac{n}{4}) - \frac{1}{2}(x - \frac{n}{3})$ to find the value of x.

SOLUTION.

$\frac{1}{4}(x - \frac{n}{5}) = \frac{1}{3}(x - \frac{n}{4}) - \frac{1}{2}(x - \frac{n}{3})$	(I)	
$3(x - \frac{n}{5}) = 4(x - \frac{n}{4}) - 6(x - \frac{n}{3})$	(II)	= (I) × 12.
$3x - \frac{3n}{5} = 4x - n - 6x + 2n$	(III)	= (II) cleared of brackets.
$3x - 4x + 6x = -n + 2n + \frac{3n}{5}$	(IV)	= (III) transposed.
$5x = n + \frac{3n}{5}$	(V)	= (IV) collected.
$25x = 5n + 3n$	(VI)	= (V) × 5.
$25x = 8n$	(VII)	= (VI) collected.
$x = \frac{8}{25}n$	(VIII)	= (VII) ÷ 25

Ex. 7. Given $\frac{a}{bx} + \frac{b}{cx} + \frac{c}{dx} + \frac{d}{fx} = g$, to find the value of x.

SOLUTION.

$\frac{a}{bx} + \frac{b}{cx} + \frac{c}{dx} + \frac{d}{fx} = g$	(I)	
$acdf + b^2df + bc^2f + bcd^2 = bcdfgx$	(II)	= (I) × $bcdfx$
$x = \frac{acdf + b^2df + bc^2f + bcd^2}{bcdfg}$	(III)	= (II) ÷ $bcdfg$

Ex. 8. Given $\frac{(a+b)x}{a-b} + \frac{x}{a^2 - b^2} = \frac{x+1}{a+b}$ to find the value of x.

SOLUTION.

$\frac{(a+b)x}{a-b} + \frac{x}{a^2 - b^2} = \frac{x+1}{a+b}$	(I)	
$(a+b)^2x + x = (a-b)(x+1)$	(II)	= (I) × $(a^2 - b^2)$
$a^2x + 2abx + b^2x + x = ax - bx + a - b$	(III)	= (II) expanded.
$a^2x + 2abx + b^2x + x - ax + bx = a - b$	(IV)	= (III) transposed.
$(a^2 + 2ab + b^2 + 1 - a + b)x = a - b$	(V)	= (IV) factored.
$x = \frac{a-b}{a^2 + 2ab + b^2 + 1 - a + b}$	(VI)	= (V) ÷ coef. of x.

EXERCISE XXXII.

Solve the following equations:—

1. $c + \frac{x}{3} = 7 - \frac{x}{4}$.

2. $2x - \frac{x}{5} = x + 4.$

3. $2x - \frac{x}{3} + \frac{x}{7} = \frac{3x - 11}{4} + x + 9.$

4. $2x - 7 + \frac{3x - 1}{5} = \frac{x + 8}{3} - 2x.$

5. $2 - \frac{x - 5}{7} = 3 - \frac{x - 7}{4}.$

6. $4x - \frac{2x}{3\frac{1}{2}} = \frac{3x + 1}{2} + x + 6.$

7. $2x - 16\frac{1}{4} = \frac{35}{4} + \frac{1}{2}x.$

8. $\frac{x + 3}{4} - \frac{x + 4}{5} - 16 = -\frac{x + 1}{3}.$

9. $4x - \frac{2x + 19}{5} = 15 - \frac{7x + 11}{4}.$

10. $\frac{7x}{9} + 3\frac{1}{3} = 21 - \frac{3\frac{1}{4}x - 7}{12}.$

11. $\frac{8x - 17}{11} + \frac{14x + 17}{13} = 3x - \frac{31 - x}{2}.$

12. $\frac{4x + 4}{3} - x = 2 + \frac{14 - 3x}{3}.$

13. $3x - \frac{4x - 5}{3} + \frac{3}{5}x = 17 + \frac{2 - 6x}{4} + \frac{3x + 1}{8}.$

14. $\frac{x}{12} + \frac{3\frac{1}{3}x - 5}{7} - \frac{2\frac{5}{6}x - 9}{5} = 1 - \frac{7\frac{1}{2}x - x + 2}{8} - \frac{9 - 5\frac{3}{4}x}{6}.$

15. $x + \frac{x}{2} + \frac{x}{4} - \frac{3(x - 7)}{5} = 2x - 2\frac{2}{5}.$

16. $21 - \frac{5(x - 1)}{8} - \frac{97 - 7x}{2} = x - \frac{1}{16}(3x - 11) - 9.$

17. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} + \frac{x}{6} = x + \frac{5x}{12} + 4.$

18. $2x - \frac{20 - x}{6} + 10 = \frac{3}{2}x$

19. $\frac{36 + 20x}{25} - \frac{1}{5}x = 3\frac{11}{25} - \frac{5x + 20}{9x - 16}.$

20. $33\frac{1}{10} - 3x - \dfrac{12 + 7x}{9} + \dfrac{9 + 5x}{10} = \dfrac{3x - 13}{16} - \dfrac{11x - 17}{8}.$

21. $\dfrac{9x + 20}{36} = \dfrac{4x - 12}{5x - 4} + \dfrac{x}{4}.$

22. $\frac{1}{2}x + \frac{1}{3}(x + 3) - \frac{1}{4}(x - 4) = \frac{1}{6}(x + 5) + 31\frac{1}{3}.$

23. $6x - \dfrac{7(x + 2)}{3} = 5 + \dfrac{2(2x + 1)}{3} - \dfrac{17 - 3x}{5}.$

24. $5x - \dfrac{2(5x^2 - 9)}{3 + 2x} = 9 - \dfrac{6x + 9}{3 + 4x}.$

25. $\dfrac{2(x + 2)}{3} - \dfrac{7x - 13}{3(1 + 2x)} = \dfrac{6x + 7}{9}.$

26. $ax + b = c.$

27. $3ax - b^2 = bc - \frac{2}{3}ax.$

28. $4bx - 3x = \frac{1}{2}(a - b^2 + 3ax).$

29. $2a^2x - \dfrac{3a - x}{b} = x - \dfrac{(a - b)x}{2a}.$

30. $3a - \dfrac{2x + a}{b} = \dfrac{4a - 3x}{c} - \dfrac{ax - b}{5}.$

31. $x + \dfrac{ax}{b} + \dfrac{cx}{d} = f.$

32. $\dfrac{bx + 4a}{4} - \dfrac{a^2 - 3bx}{a} - bx = ab^2 - \dfrac{5a^2 - 6bx}{2a} + ax.$

33. $ax - bc = \dfrac{b^2x}{b - a}.$

34. $\dfrac{11a - 3x}{a + b} - \dfrac{6a - 5x}{a - b} = \dfrac{a + b}{a - b} + \dfrac{2x}{a^2 - b^2}.$

35. $\dfrac{(a + x)^2}{4} - abx = \frac{1}{4}x^2.$

36. $\dfrac{abc}{\frac{1}{3}(a+b)} - \dfrac{bx}{a} + \dfrac{a^2b^2}{(a + b)^3} = 3cx - \dfrac{b^2x}{a} \cdot \dfrac{2a + b}{(a+b)^2}.$

37. $3 + 1{\cdot}72x - 2{\cdot}21x = {\cdot}203x.$

38. ${\cdot}\dot{3}x + x(6 - a) = 3a - {\cdot}\dot{2}\dot{3}x.$

39. $\frac{5}{6}(x - \frac{1}{3}) + \frac{1}{3}\{1 - (x + \frac{2}{5})\} - \frac{2}{7}\{x - (1 + \frac{1}{3}x)\} = x + \frac{2}{7}x.$

40. $\dfrac{8ax - b}{5} - \dfrac{5b}{3} = 4 - b - \dfrac{7c}{9}.$

41. $(a^2 - x)(b^2 + x) - 3ab(1 - x) = (x - a)(c - x).$

PROBLEMS

PRODUCING SIMPLE EQUATIONS INVOLVING ONLY ONE UNKNOWN QUANTITY.

120. A Problem is a written statement of the relations existing between certain quantities whose values are given, and another quantity or other quantities whose values are to be found. The solution of problems consists of two distinct parts:

I. The *Algebraic Statement*, or briefly the *statement*. This consists in the translation of the problem into *algebraic language*, i. e., in expressing the conditions of the problem, the relations between the given and the unknown quantities, by means of signs and symbols, so as to *indicate* the *operations described* in the problem.

II. The solution of the resulting equation.

121. It is with the former of these parts, i. e., "the statement," that the student experiences the chief difficulty, the nature of problems being such that they admit of no general rule for their statement. The student must, therefore, be left very much to his own ingenuity, and he can expect to acquire facility in the operation only by long continued practice. He will, however, be very much assisted in his efforts by attention to the following general instructions for making:—

THE STATEMENT OF PROBLEMS.

I. *Read over the problem carefully, until its conditions are clearly apprehended, and it is distinctly understood what is given and what is required.*

II. *Represent the unknown quantity by x, and set down in algebraic language the relations existing between it and the given quantities, as described in the problem, or in other words, indicate upon x, by means of signs, the same operation that would be necessary to verify its value in the equation if that value were already determined.*

NOTE.—Before commencing the exercise the beginner is particularly directed to *study carefully* the solution of the preliminary problems, in order to observe the modes of proceeding to make the statement.

Ex. 1. What number is that from the double of which if 10 be subtracted the remainder is 44 ?

SOLUTION.

Here we have given that a certain number is such that when its double is diminished by 10 the remainder is 44.

Let x = the number.

Then $2x$ = its double, and $2x - 10$ = its double diminished by 10.

Then, by the problem, $2x - 10 = 44$, which is the required statement.

$$2x = 54, \text{ by transposition.}$$
$$x = 27, \text{ by division.}$$

Therefore 27 is the number required.

Verification. $(27 \times 2) - 10 = 44$
$$54 - 10 = 44$$
$$44 = 44$$

Ex. 2. Find a number such that one-half, one-third, and one-fourth of it added together shall exceed the number itself by $4\frac{1}{2}$.

SOLUTION.

Here we have given that $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ of a certain number $>$ the number itself by $4\frac{1}{2}$, or what amounts to the same thing, that $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ of a certain number = the number itself $+ 4\frac{1}{2}$.

Let x = the number; then $\frac{x}{2} = \frac{1}{2}$ of it; $\frac{x}{3} = \frac{1}{3}$ of it; and $\frac{x}{4} = \frac{1}{4}$ of it.

And $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = x + 4\frac{1}{2}$, which is the statement required.

$$6x + 4x + 3x = 12x + 54 \qquad (\text{II}) = (\text{I}) \times 12.$$
$$6x + 4x + 3x - 12x = 54 \qquad (\text{III}) = (\text{II}) \text{ transposed.}$$
$$x = 54 \qquad (\text{IV}) = (\text{III}) \text{ collected.}$$

Therefore 54 is the required number.

Verification. $\frac{54}{2} + \frac{54}{3} + \frac{54}{4} = 54 + 4\frac{1}{2}$
$$27 + 18 + 13\frac{1}{2} = 58\frac{1}{2}$$
$$58\frac{1}{2} = 58$$

Ex. 3. Divide the number 112 into two such parts that if 21 be added to the less the sum shall be less than one-third of the greater by the third part of unity.

SOLUTION.

Here 112 is to be divided into two parts such that the less + 21 shall be equal to ($\frac{1}{3}$ of the greater) $- \frac{1}{3}$.

Let x = the greater part; then since 112 is the sum of the two parts, $112 - x$ = the less.

$(112 - x) + 21$ is 21 added to the less, and $\frac{x}{3} - \frac{1}{3}$ is $\frac{1}{3}$ of unity less than $\frac{1}{3}$ of greater.

Then $(112 - x) + 21 = \frac{x}{3} - \frac{1}{3}$, which is the statement.

$$336 - 3x + 63 = x - 1 \quad \text{(II)} = \text{(I)} \times 3$$

$$-3x - x = -1 - 63 - 336 \quad \text{(III)} = \text{(II) transposed.}$$

$$-4x = -400 \quad \text{(IV)} = \text{(III) collected.}$$

$$x = 100 = \text{greater.}$$

$$112 - x = 112 - 100 = 12 = \text{less.}$$

Verification. $(112 - 100) + 21 = \frac{100}{3} - \frac{1}{3}$

$$112 - 100 + 21 = \frac{100}{3} - \frac{1}{3}$$

$$133 - 100 = 33\tfrac{1}{3} - \tfrac{1}{3}$$

$$33 = 33$$

Ex. 4. What sum of money is that from which if \$46·20 be subtracted, one-half the remainder shall exceed one-third of the remainder by \$50.

SOLUTION.

Here the sum of money is such that

$\frac{1}{2}$(Sum − \$46·20) is $>$ by \$50 than $\frac{1}{3}$ (Sum − \$46·20).

Let x = the sum of money.

Then x − \$46·20 is \$46·20 subtracted from the sum.

$\frac{x - \$46\cdot20}{2}$ is half the rem., and $\frac{x - \$46\cdot20}{3}$ is one-third of rem.

$$\text{Then } \frac{x - \$46\cdot20}{2} - \$50 = \frac{x - \$46\cdot20}{3} \quad \text{(I)}.$$

$$3x - \$138\cdot60 - \$300 = 2x - \$92\cdot40 \quad \text{(II)} = \text{(I)} \times 6.$$

$$3x - 2x = -\$92\cdot40 + 138\cdot60 + \$300.$$

$$x = \$346\cdot20 = \text{sum required.}$$

NOTE.—The student should verify the result in every case, as is done in the three preceding problems.

Ex. 5. A certain number consists of two digits, such that the right hand digit exceeds the left hand digit by 2; and if the sum of the digits be increased by $\frac{9}{7}$ of the number, the digits will be inverted. Required the number.

SOLUTION.

Let x = the left hand digit.

Then $x + 2$ = the right hand digit.

$10x + (x + 2)$ = the number.*

$x + x + 2$ = the sum of the digits.

$2x + 2 + \frac{9}{7}(10x + x + 2)$ = the sum of the digits increased by $\frac{9}{7}$ of the number.

$10(x + 2) + x$ = number with its digits inverted.

$$\begin{aligned} \text{Then } 2x + 2 + \tfrac{9}{7}(10x + x + 2) &= 10(x + 2) + x. \\ 14x + 14 + 9(11x + 2) &= 70(x + 2) + 7x. \\ 14x + 14 + 99x + 18 &= 70x + 140 + 7x. \\ 99x + 14x - 70x - 7x &= 140 - 14 - 18. \\ 36x &= 108. \\ x &= 3 = \text{left hand digit.} \\ x + 2 &= 5 = \text{right hand digit.} \end{aligned}$$

Therefore the number is 35.

Ex. 6. A can do a piece of work in 10 days, which A and B can together finish in 6 days. In what time can B working alone do the work?

SOLUTION.

Let x = number of days B would require to do the work.

Since A does whole work in 10 days, in 1 day he would do $\frac{1}{10}$ of it.

Since B does whole work in x days, in 1 day he would do $\frac{1}{x}$ of it.

*Note.—If we take any number, as 6542, and represent its digits respectively by the letters d, c, b, and a, then $d + c + b + a$ will express, not the number, but merely the sum of its digits. In order to express the number we must take into account the local as well as the absolute values of the digits, i.e., we must remember that the first digit being so many units, the second is so many tens, the third so many hundreds, &c.

Hence $d + c + b + a = 6 + 5 + 4 + 2 = 17$ = sum of digits.

And $1000d + 100c + 10b + a = 6000 + 500 + 40 + 2 = 6542$ = the number.

And of course $1000a + 100b + 10c + d = 2000 + 400 + 50 + 6 = 2456$ = number with its digits inverted.

Since A and B do the work in 6 days, in 1 day they would do $\frac{1}{6}$ of it.

Then A's work for 1 day + B's work for 1 day = work of both A and B for 1 day.

That is, $\frac{1}{10} + \frac{1}{x} = \frac{1}{6}$ (I).

$$3x + 30 = 5x \quad \text{(II)} = \text{(I)} \times 30x \text{ to clear of fractions.}$$
$$3x - 5x = -30 \quad \text{(III)} = \text{(II) transposed.}$$
$$-2x = -30 \quad \text{(IV)} = \text{(III) collected.}$$
$$x = 15 = \text{days } B \text{ would require.}$$

Ex. 7. A person being asked how many ducks and geese he had, replied that if he had 8 more of each he would have 7 geese for 8 ducks, but that if he had 8 less of each he would only have 6 geese for 7 ducks. How many had he of each?

SOLUTION.

Let x = the number of ducks he had.

Then $x + 8$ = number of ducks increased by 8.

$\frac{x+8}{8}$ = number of times he had 7 geese.

$\frac{x+8}{8} \times 7$ = number of geese he had when increased by 8.

Hence number of geese = 8 less than $\frac{x+8}{8} \times 7 = \frac{7}{8}(x+8) - 8$.

Also $x - 8$ = number of ducks diminished by 8.

$\frac{7}{8}(x+8) - 16$ = number of geese diminished by 8; and by the question, $\frac{x-8}{7} = \frac{\frac{7}{8}(x+8) - 16}{6}$.

$$6(x-8) = 7\left\{\tfrac{7}{8}(x+8) - 16\right\}.$$
$$6x - 48 = \tfrac{49}{8}(x+8) - 112.$$
$$6x + 64 = \frac{49x + 392}{8}.$$
$$48x + 512 = 49x + 392.$$
$$x = 120 = \text{number of ducks.}$$

$\frac{7}{8}(120 + 8) - 8 = (\frac{7}{8} \text{ of } 128) - 8 = (7 \times 16) - 8 = 112 - 8 = 104$ = number of geese.

Ex. 8. A merchant has tea worth 4s. 3d. and 5s. 9d. per lb. How many lbs. of each must there be in a chest of 126 lbs., which shall be worth £30 ?

SOLUTION.

Let x = number of lbs. at 4s. 3d. or 17 threepences per lb.

Then $120 - x$ = number of lbs. at 5s. 9d. or 23 threepences per lb.

$17x$ = worth in threepences of x lbs. at 4s. 3d. per lb.

$23(120 - x)$ = worth in threepences of $120 - x$ lbs. at 5s. 9d. per lb.

2400 = number of threepences in £30.

$$\begin{aligned}
\text{Then } 17x + 23(120 - x) &= 2400. \\
17x + 2760 - 23x &= 2400. \\
17x - 23x &= 2400 - 2760. \\
-6x &= -360. \\
x &= 60 = \text{lbs. at 4s. 3d. per lb.} \\
120 - 60 &= 60 = \text{lbs. at 5s. 9d. per lb.}
\end{aligned}$$

Ex. 9. Divide the number 90 into four parts such that the first increased by 2, the second diminished by 2, the third divided by 2, and the fourth multiplied by 2, shall all be equal to the same quantity.

SOLUTION.

Let x = the quantity to which the 1st part is equal when increased by 2.

Then $x - 2$ = 1st part ; $x + 2$ = 2nd part ;

$x \times 2$ = 3rd part ; $x \div 2$ = 4th part.

$$\begin{aligned}
\text{Then } (x - 2) + (x + 2) + 2x + \tfrac{x}{2} &= 90. \\
x - 2 + x + 2 + 2x + \tfrac{x}{2} &= 90. \\
4x + \tfrac{x}{2} &= 90. \\
8x + x &= 180. \\
9x &= 180. \\
x &= 20.
\end{aligned}$$

$x - 2 = 20 - 2 = 18$ = 1st part ; $x + 2 = 20 + 2 = 22$ = 2nd part.

$2x = 20 \times 2 = 40$ = 3rd part ; $\frac{x}{2} = \frac{20}{2} = 10$ = 4th part.

Ex. 10. A workman is engaged for n days, at p cents per day, upon condition that for every day that he is idle instead of receiving anything he shall forfeit q cents. At the end of the time agreed upon he received c cents. Required the number of days he worked, and the number of days he was idle.

SOLUTION.

Let x = the number of days he worked.

Then $n - x$ = the number of days on which he was idle.

px = number of cents he received for x days work.

$q(n-x)$ = number of cents he forfeited for $(n-x)$ days idleness.

Then $px - q(n-x) = c$.

$$px - qn + qx = c.$$

$$px + qx = c + qn.$$

$$(p+q)x = c + qn.$$

$$x = \frac{c+qn}{p+q} = \text{number of working days.}$$

$$n - \frac{c+qn}{p+q} = \frac{np + nq - c - nq}{p+q} = \frac{np - c}{p+q} = \text{number of idle days}$$

EXERCISE XXXIII.

1. Required two numbers whose sum is 47 and difference 13.

2. There are two numbers, one of which is greater than the other by 21, and the quotient of their sum by the less is 3; what are the numbers?

3. After paying away $\frac{2}{3}$ and $\frac{2}{7}$ of my money, I had \$2·50 remaining; how much had I at first?

4. Find a number such that if 21 be taken from it, and the remainder divided by $8\frac{3}{5}$, the quotient will be 5.

5. Divide 54 into three such parts that the first divided by 2, the second by 3, and the third by 4, shall all give the same quotient.

6. Paid $\frac{2}{5}$ of my debts, and then paid $\frac{3}{7}$ of the remainder, and afterwards owe \$192; how much did I owe at first?

7. A drove of cattle is disposed of as follows: $\frac{1}{3}$ to A, $\frac{1}{6}$ to B, $\frac{1}{5}$ to C, and the remainder, which was 9, to D; how many cattle was there in the drove?

8. A farmer has two flocks of sheep, each containing the same number; but when he had sold 19 sheep from one flock and 91 from the other, the former now contained twice as many as the latter. Required the number originally in each flock.

9. Find a number whose fourth part exceeds its seventh part by 6.

10. What number is that the double of which exceeds $\frac{2}{7}$ of its half by 25.

11. Find a number such that increased by one-half of itself the sum shall be 39.

12. What number is that which exceeds the sum of its half and its third parts by 17?

13. Find a number such that when 15 is taken from its double, and to half the remainder 7 is added, the sum is greater by 3 than $\frac{3}{4}$ of the original number.

14. What number is that to which if 11 be added, two and a-half times the sum shall be 85.

15. Find a number such that one-half, two-thirds, and three-fourths of it added together, shall exceed $1\frac{3}{4}$ times the original number by 21.

16. A farmer sold a load containing a certain number of barrels of apples for $36, and he afterwards sold a second load at the same rate, but as it contained 5 barrels less than the former, he only received $21. What was the price per barrel, and what was the number of barrels in each load?

17. A person starts to walk from Toronto to Brampton at the rate of $3\frac{1}{2}$ miles per hour; precisely $28\frac{1}{2}$ minutes afterwards another person starts from Brampton to walk to Toronto at the rate of 4 miles per hour, and they meet one another exactly half-way between the two places. Required the distance from Toronto to Brampton.

18. In a certain grist-mill there are three runs of stones; the first of which can empty the granary in 72 hours, the second in 84 hours, and the third in 90 hours. Two teams are engaged drawing wheat and storing it in the granary, and of these the first can fill it in 60 hours, and the second in 78 hours. Now if the granary be full, and both teams and all three runs of stones be set in operation, in what time will it be emptied?

19. If from the number of the year in which all the slaves in Canada received their freedom, the number 1780 be taken, three times the remainder increased by 1620, will give the year of the celebrated Indian massacre of Lachine, and if the two dates be added together, one-half their sum increased by 116 will give the year 1862. Required the date of the abolition of slavery in Canada, and also that of the massacre of Lachine?

20. Divide \$7400 among A, B, and C, so that A shall have \$120 more than B; and C \$106 less than A.

21. A pupil receives 24 music lessons and 32 drawing lessons in the quarter, and the former cost her \$3 more than the latter; if, however, she had received 32 music lessons and only 24 drawing lessons, the latter would have cost her, at the same rate, \$10 less than the former. Required the price per lesson for music and drawing?

22. A library contains twice as many volumes on General Literature as on History, $1\frac{1}{2}$ times as many volumes on History as on Biography, as many volumes on Biography as on Travels, and three times as many volumes on Travels as on the Sciences, and the number of volumes on the Sciences is 70. Required the number of volumes in the library?

23. The Rideau Canal is six miles less than four times as long as the Niagara River, and their combined length doubled and decreased by 100 miles, exceeds the length of the Great Western Railway by one mile. The G. W. R. being 229 miles long, required the length of the Rideau Canal, and also that of the Niagara River?

24. A can do a piece of work in 12 days, which B can finish in 15 days, and C in 18 days. Now A and B work together at it for 1 day; B and C work together at it for two days: in what time will all three finish the work remaining to be done?

25. Divide a number n into two parts, such that one may exceed the other by $(a - c)$.

26. What is the first hour after 12 o'clock at which the two hands of a watch are (I) together, (II) directly opposite, and (III) at right angles to one another?

27. A man owns two fields and a horse, the latter being worth \$90. He offers to sell the first field with the horse in it for \$25

more than he asks for the second field alone, but for the second field with the horse in it he asks double as much as for the first field alone. Required the price of each field?

28. A, B, and C can do a piece of work in 20 days, which A can do alone in 50, and B alone in 65 days. C works at it for 11 days, then B and C together for 5 days. In what time can A and C finish the remainder?

29. Divide \$7189 among A, B, C, and D, so as to give to A as much as the other three, to B \$40 more than two-fifths of the shares of C and D; and to D \$25·40 less than three-sevenths of C's share.

30. A piece of work can be finished by 4 men in 9 days, or by 10 women in 7 days, or by 15 children in 8 days. In what time can 1 man, 3 women, and 4 children finish the work?

31. There is a number consisting of two digits, whose sum is 14 (the right hand digit being the greater), and three-seventeenths of the number is equal to three halves of the right hand digit. Required the number?

32. A farmer sold his farm for \$8600, and considered that he had cleared a certain amount by the transaction. A note, however, for \$640, which he had accepted in part payment, turned out to be worthless, and he found that, in consequence, he lost upon the whole transaction two-fifths as much as he would have gained had the note been good. What was the value of the property?

33. There is a fish whose tail weighs 9 lbs., his head weighs as much as his tail and half his body, and his body weighs as much as his head and tail together. What is the weight of the fish?

34. A merchant yearly increases his capital by one-third of itself, but takes away \$1000 for current expenses. At the end of the third year after taking away the \$1000 he finds that the original capital was doubled. What was his capital at starting?

35. The fore-wheel of a waggon is a feet; and the hind-wheel b feet in circumference; through what distance must the waggon pass in order that the fore-wheel shall have made n revolutions more than the hind-wheel?

36. The hour and minute-hands of a watch are together at

noon. When and how often will they be together during the next twelve hours?

37. Divide the number 96 into two such parts that when the greater is divided by 7 and the less multiplied by 3, the sum of the quotient and product shall be 30.

38. Divide $2560 among A, B, and C, so that A shall have half as much again as B; and that C shall have half as much again as A.

39. A steamer makes the *down* trip from the head of Lake Ontario to Montreal in 28 hours, the current being in its favor. When returning it is found that in ascending the St. Lawrence (three-sevenths of the entire trip) the rate of sailing is 5 miles per hour less than the average rate in its downward journey, but upon entering the lake it is enabled to increase its speed 2 miles per hour, and again reaches Hamilton, at the head of the Lake, in $\frac{19}{21}$ of the time it would have required had the rate been uniformly the same as when ascending the river. Required the distance between Montreal and Hamilton, and the rates of sailing?

40. A gentleman bequeaths his property as follows:—To his eldest child he leaves $1800 and $\frac{1}{6}$ of the rest of his property; to the second twice $1800 and $\frac{1}{6}$ of the part now remaining; to the third three times $1800 and $\frac{1}{6}$ of the part now remaining, and so on. By this arrangement his property is divided equally among his children. How many children were there, and what was the fortune of each?

41. A certain number consists of two digits, whose difference is 7—the right hand one being the greater. When the number is divided by the sum of its digits it gives a quotient 2, with a remainder 7. Find the number.

42. Divide $2100 among A, B, and C, so that A shall have $80 more than $\frac{3}{5}$ of B and C's shares together, and that C shall have $20 less than B.

43. A nurserymen has an orchard to plant with a given number of trees, and he finds that when he has as many rows as trees in a row there are 75 trees remaining, but if he puts 5 trees less in a row, and increases the number of rows by 6, he then has only 5 trees remaining. What was the number of trees?

44. Divide the number a into two such parts that the one shall be $\frac{n}{m}$ths of the other?

45. What are the two parts of 60 such that their product is equal to three times the square of the less?

46. Twelve oxen are turned into a field of grass containing $3\frac{1}{3}$ acres, and by the end of 4 weeks have not only eaten all the grass on it when they were turned in, but also all that grew during the 4 weeks. Similarly in 9 weeks 21 oxen eat all the grass that grows on 10 acres during that time, together with what was on the field when they were turned in. Now assuming in all cases that the original quantity and quality per acre, and the growth per acre, is the same, how many oxen can in this way graze for 18 weeks on 24 acres?

47. Divide the number a into three parts such that the second may be n times and the third m times as great as the first.

48. Divide the number a into three parts such that the second shall be m times the nth part of the first, and that the third shall be the qth part of p times the first.

49. From the first of two mortars in a battery 36 shells are thrown before the second is ready for firing. Shells are then thrown from both in the proportion of 8 from the first to 7 from the second; the second mortar requiring as much powder for 3 charges as the first does for 4. How many balls must the second mortar throw in order that both may have consumed the same quantity of powder?

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE, INVOLVING ONLY TWO UNKNOWN QUANTITIES.

122. For the solution of equations involving two or more unknown quantities, as many *independent* equations are required as there are unknown quantities involved.

Thus, the equation $x + y = 8$ is called an *indeterminate* equation, because an unlimited number of values may be assigned to x and y, so as to satisfy the equation. For example, we may take $x = \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 0$, &c., and $y = 7\frac{3}{4}, 7\frac{1}{2}, 7, 6, 5, 4, 3, 2, 1, 0, 8$, &c., and the equation will be satisfied by any pair of these values.

But if we take the equation $x + y = 8$, and limit it by another corres-

ponding but independent equation, as for example, $2x - 3y = 1$, we shall find that the two equations are only satisfied by the value $x = 5$ and $y = 3$. An equation of this kind is called a *determinate* equation.

123. A set of two or more equations thus mutually limiting the values of the unknown quantities involved, form what is called a *simultaneous* equation.

124. As stated in Art. 122, in order that the equation may be determinate, there must be as many *independent* equations as there are unknown quantities involved. Now *equations are said to be independent when they express different relations between the unknown quantities.*

NOTE.—That is, the two or three equations given must not be derived from one another by mere multiplication, or division, or subtraction, or addition. Thus, if $x + y = 8$ be one of the equations, it would be useless to associate with it $2x + 2y = 16$, or $\frac{1}{7}x + \frac{1}{7}y = 1\frac{1}{7}$, or $x + 2y = 8 + y$, $y - 3x = 8 - 4x$, &c., because these equations, though true in themselves, express no new relation between the unknown quantities, and are all reducible to the form of $x + y = 8$, having obviously been derived from it by mere addition, subtraction, multiplication, or division.

125. Simultaneous equations are solved by *elimination*, as it is termed, i. e., by so combining the given equations as to get rid of one of the unknown quantities, and thus to obtain from them a new equation involving only one unknown.

126. There are three methods of eliminating one of the unknown quantities, and thus of solving simultaneous equations.

ELIMINATION BY ADDITION OR SUBTRACTION.

RULE.

127. I. *If the coefficients of the quantity we desire to eliminate are not already the same in both equations, multiply one or both equations by such multipliers as shall make the coefficients of that quantity similar.*

II. *Having thus prepared the two equations, add them, member to member, if the signs of the quantity to be eliminated are unlike; subtract one equation from the other if the signs in question are like.*

Ex. 1. Given $\left.\begin{aligned} 4x - 3y &= 6 \\ 4x + 7y &= 26 \end{aligned}\right\}$ to find the values of x and y.

SOLUTION.

$4x - 3y = 6$	(I)	Here as the coef. of x is the same in both equations there is no necessity of multiplying, and we accordingly subtract at once.
$4x + 7y = 26$	(II)	
$10y = 20$	(III)	= (II) − (I).
$y = 2$	(IV)	= (III) ÷ 10.
Then $4x - 3y = 4x - 6 = 6$	(V)	= (I) by substituting 2 for y.
$4x = 12$		
$x = 3$		

Therefore values are $x = 3$ and $y = 2$.

Ex. 2. Given $\left.\begin{aligned} 4x + 3y &= 43 \\ 3x - 2y &= 11 \end{aligned}\right\}$ to find the values of x and y.

SOLUTION.

$4x + 3y = 43$	(I)	
$3x - 2y = 11$	(II)	
$8x + 6y = 86$	(III)	= (I) × 2.
$9x - 6y = 33$	(IV)	= (II) × 3.
$17x = 119$	(V)	= (III) + (IV). We add because the signs are unlike.
$x = 7$	(VI)	= (V) ÷ 17.
$4x+3y = 28+3y = 43$	(VII)	= (I) with 7 substituted for x.
$3y = 15$		
$y = 5$		

Therefore values are $x = 7$ and $y = 5$.

NOTE.—We can always prepare the equations for addition or subtraction by multiplying each by that coef. of the unknown to be eliminated, which is given in the other equations. Sometimes, however, it is not necessary to multiply *both* equations, but we can find by inspection a multiplier for one only, which will at once prepare the equation for elimination.

Thus, if $\left.\begin{aligned} 4x - 3y &= 8 \\ 2x + 9y &= 46 \end{aligned}\right\}$ be the equation as given and we wish to eliminate x, we may multiply the lower equation by 4 and the upper by 2, and then subtract, but we may obviously attain the same end, in the elimination of x, by simply multiplying the lower equation by 2, and then subtracting. Similarly if we wish to eliminate the y, instead of multiplying the upper equation by 9, and the lower by 3, we may prepare the two equations for addition by simply multiplying the upper by 3.

Ex. 3. Given $\left.\begin{aligned} ax + y &= m \\ bx - ay &= n \end{aligned}\right\}$ to find the values of x and y.

SOLUTION.

$ax + y = m$	(I)	
$bx - ay = n$	(II)	
$a^2x + ay = am$	(III)	$=$ (I) $\times a$.
$a^2x + bx = am + n$	(IV)	$=$ (II) + (III).
$(a^2 + b)x = am + n$	(V)	$=$ (IV) factored.
$x = \dfrac{am + n}{a^2 + b}$	(VI)	$=$ (V) $\div a^2 + b$.
$ax + y = \dfrac{am + n}{a^2 + b} \times a + y = m$	(VII)	$=$ (I) with value of x substituted for x.

$$y = m - \frac{a^2m + an}{a^2 + b}.$$

$$y = \frac{a^2m + bm - a^2m - an}{a^2 + b} = \frac{bm - an}{a^2 + b}.$$

ELIMINATION BY SUBSTITUTION.

Rule.

128. I. *Find from one of the given equations the value of the unknown to be eliminated in terms of the other unknown quantity.*

II. *Substitute this value in the remaining equation for the same unknown quantity, and there will result an equation containing only one unknown quantity.*

Ex. 4. Given $\left.\begin{aligned} 2x - y &= 1 \\ 7x + 9y &= 16 \end{aligned}\right\}$ to find the values of x and y.

SOLUTION.

$2x - y = 1$	(I)	
$7x + 9y = 16$	(II)	
$y = 2x - 1$	(III)	$=$ (I) transposed.
$7x + 9(2x - 1) = 16$	(IV)	$=$ (II) with $2x - 1$ substituted for y.
$7x + 18x - 9 = 16$	(V)	$=$ (IV) expanded.
$25x = 25$	(VI)	$=$ (V) transposed and collected.
$x = 1$	(VII)	$=$ (VI) $\div 25$.
$y = 2x - 1 = 2 - 1 = 1$	(VIII)	$=$ (III) with value of x substituted.

Ex. 5. Given $\left.\begin{array}{l} 5x - \dfrac{4y - 7x}{6} = 8 \\ 7x - \dfrac{4y}{11} + \dfrac{7x - 2y}{6} = 3y - 8 \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$5x - \dfrac{4y + 7x}{6} = 8$	(I)	
$7x - \dfrac{4y}{11} + \dfrac{7x - 2y}{6} = 3y - 8$	(II)	
$23x - 4y = 48$	(III)	= (I) reduced.
$539x - 244y = -528$	(IV)	= (II) reduced.
$x = \dfrac{48 + 4y}{23}$	(V)	= (III) transp. and $\div$ 23.
$539\left(\dfrac{48 + 4y}{23}\right) - 244y = -528$	(VI)	= (IV) with $\dfrac{48 + 4y}{23}$ sub. for x
$\dfrac{25872 + 2156y}{23} - 244y = -528$	(VII)	= (VI) expanded.
$3456y = 38016$	(VIII)	= (VII) reduced.
$y = 11$	(IX)	= (VIII) $\div$ 3456.
$x = \dfrac{48 + 4y}{23} = \dfrac{48 + 44}{23} = 4$	(X)	= (V) with 11 substitut. for y.

Therefore the required values are $x = 4$ and $y = 11$.

ELIMINATION BY COMPARISON.

RULE.

129. I. *Find from the first equation the value of the quantity to be eliminated, in terms of the other unknown quantity; and similarly find another value for the same quantity from the second equation.*

II. *Place these values equal to one another, i. e., form an equation by placing the sign of equality between them.*

Ex. 6. Given $\left.\begin{array}{r} x + 64y = 1552 \\ 64x + y = 1048 \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$x + 64y = 1552$	(I)	
$64x + y = 1048$	(II)	
$\therefore x = 1552 - 64y$	(III)	= (I) transposed.
$x = \dfrac{1048 - y}{64}$	(IV)	= (II) transp. and $\div$ 64.
$\therefore \dfrac{1048 - y}{64} = 1552 - 64y$	(V)	$\because$ first members of (III) and (IV) are = $\therefore$ also the second members are = (Ax. XI).
$1048 - y = 99328 - 4096y$	(VI)	= (II) $\times$ 64 to clear of fractions.
$4095y = 98280$	(VII)	= (VI) transposed and collected.
$y = 24$	(VIII)	= (VII) $\div$ 4095.
$x = 1552 - 64y = 1552 - 1536 = 16$	(IX)	= (III) with 24 substituted for y.

NOTE.—*Although either of these three methods may be employed, the student is recommended, as a rule, to use the first, that being upon the whole the most convenient.*

EXERCISE XXXIV.

Find the values of x and y in the following equations :—

1. $7x - 3y = 5$; and $4x + y = 11$.
2. $x + 3y = 23$; and $6x - y = 24$.
3. $3x - 11y = 1$; and $5x - 7y = 64$.
4. $5x + 6y = 80$; and $9x - 5y = -14$.
5. $\frac{1}{7}x + y = 4$; and $4x - \frac{1}{3}y = 27$.
6. $\frac{3}{8}x - \frac{2}{3}y = -11$; and $\frac{2}{3}x + \frac{7}{16}y = 37$.
7. $11x + y + 11 = 59 - \dfrac{2y + 9x}{2} + \dfrac{3x}{2}$; and $11 - \dfrac{7x + 13y}{3} = y - x - \dfrac{8x - 3y}{4} - (x + y + \frac{1}{2})$.
8. $\frac{1}{2}(x + 3y) - \frac{1}{3}(x - 2y) = 2$; and $\frac{1}{6}(x - y) + \frac{2}{3}(x + 5y) = 10$.
9. $19x + 18y = 147$; and $17(x + y) - 16(x - y) = 168$.
10. $2x + 3y = a$; and $5x - 2y = b$.
11. $3x + ay = m$; and $4x + by = n$.

12. $ax - 2ay = b$; and $2bx - by = c$.

13. $x - y = a$; and $x^2 - y^2 = b$.

14. $\frac{x}{a} - \frac{y}{c} = m$; and $\frac{x+y}{c} - \frac{x-y}{m} = n$

15. $\frac{m}{x} + \frac{n}{y} = a$; and $\frac{b}{x} - \frac{q}{y} = b$.

16. $x + y = 11$; and $x^2 - y^2 = 55$.

17. $\frac{\frac{1}{3}(45x + 4y)}{33} + 2 = y + 1 - \frac{1}{8}(3y + x - 3)$; and $\frac{3x + 2y}{6} - \frac{y - 5}{4}$
$= \frac{11x + 152}{12} - \frac{3y + 1}{2}$.

18. $\frac{x}{a} - \frac{y}{c} = p$; and $\frac{c}{a - x} + \frac{a}{c + y} = 0$.

19. $\frac{x - 6}{7y} + \frac{4x + 7}{24} - \frac{\frac{1}{7}(7x - y)}{6} = \frac{19 + y}{42} - \frac{\frac{1}{3}(11x + 18)}{56y}$; and
$\frac{12x - 15y + \frac{13}{4}}{10y - 8x + \frac{86}{3}} = \frac{93 - 9x}{6x - \frac{14}{6}}$.

20. $3x + 5y = \frac{(8a - 2b)ab}{a^2 - b^2}$; and $a^2x - \frac{ab^2c}{a + b} + (a + b + c)by$
$= b^2x + (a + 2b)ab$.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE,

INVOLVING MORE THAN TWO UNKNOWN QUANTITIES.

130. If we have three equations involving three unknown quantities, we may obtain their values by the following:—

RULE.—*Combine by Arts.* 127, 128, 129, *the first and second of the given equations, so as to eliminate one of the unknown quantities. Also combine the first and third, or the second and third, so as to eliminate the same unknown quantity. There will result from this process two equations involving but two unknown quantities the values of which may be obtained by the previous rules.*

EX. Given $2x + 4y - 3z = 22$; and $4x - 2y + 5z = 18$, and $6x + 7y - z = 63$, to find the values of x, y, and z.

SOLUTION.

$2x + 4y - 3z = 22$	(I)	
$4x - 2y + 5z = 18$	(II)	
$6x + 7y - z = 63$	(III)	
$4x + 8y - 6z = 44$	(IV)	= (I) × 2.
$10y - 11z = 26$	(V)	= (IV) − (II).
$6x + 12y - 9z = 66$	(VI)	= (I) × 3.
$5y - 8z = 3$	(VII)	= (VI) − (III).
$10y - 11z = 26$	(V)	
$5y - 8z = 3$	(VII)	
$10y - 16z = 6$	(VIII)	= (VII) × 2.
$5z = 20$	(IX)	= (V) − (VIII).
$z = 4$	(X)	= (IX) ÷ 5.
$5y - 8z = 5y - 32 = 3$	(XI)	= (VII) with 4 for z.
$5y = 35$	(XII)	= (XI) transposed.
$y = 7$	(XIII)	= (XII) ÷ 5.
$2x + 4y - 3z = 2x + 28 - 12 = 22$	(XIV)	= (XIII) with 4 substituted for z and 7 for y.
$2x = 6$	(XV)	= (XIV) transposed.
$x = 3$	(XVI)	= (XV) ÷ 2.

131. When there are more than three unknown quantities, and consequently more than three equations, we proceed in a similar manner, so that for solving a set of n equations involving n unknown quantities, we use the following:—

RULE.

I. *Combine one of the given* n *equations with each of the others separately, eliminating the same unknown quantity; there will result* n − 1 *equations, involving* n − 1 *unknown quantities.*

II. *Combine one of these equations with each of the others separately, eliminating a second unknown quantity; there will result* n − 2 *equations involving only* n − 2 *unknown quantities.*

III. *Continue thus combining and eliminating until an equation is obtained involving only one unknown quantity.*

IV. *Having solved this equation and thus found the value of one unknown quantity, substitute this value in one of two preceding equations, and thus obtain the value of a second unknown quantity; then substitute the values of these two unknown quantities in one of the three equations which involve only three unknowns, and thus determine the value of another, and so on, until all the values are found.*

Ex. Given $\left.\begin{aligned} v + x + y + z &= 14 \\ 3v - 2x + 4y - 3z &= 5 \\ 2v - 5x + 2y + 4z &= 24 \\ 4v + 3x - 3y - 2z &= 3 \end{aligned}\right\}$ to find the values of v, x, y, and z.

SOLUTION.

$v + x + y + z = 14$	(I)	
$3v - 2x + 4y - 3z = 5$	(II)	
$2v - 5x + 2y + 4z = 24$	(III)	
$4v + 3x - 3y - 2z = 3$	(IV)	
$3v + 3x + 3y + 3z = 42$	(V)	$=$ (I) $\times 3$.
$2v + 2x + 2y + 2z = 28$	(VI)	$=$ (I) $\times 2$.
$4v + 4x + 4y + 4z = 56$	(VII)	$=$ (I) $\times 4$.
$5x - y + 6z = 37$	(VIII)	$=$ (V) $-$ (II).
$7x - 2z = 4$	(IX)	$=$ (VI) $-$ (III).
$x + 7y + 6z = 53$	(X)	$=$ (VII) $-$ (IV).
$35x - 7y + 42z = 259$	(XI)	$=$ (VIII) $\times 7$.
$36x + 48z = 312$	(XII)	$=$ (X) $+$ (XI).
$3x + 4z = 26$	(XIII)	$=$ (XII) $\div 12$.
$14x - 4z = 8$	(XIV)	$=$ (IX) $\times 2$.
$17x = 34$	(XV)	$=$ (XIII) $+$ (XIV).
$x = 2$	(XVI)	$=$ (XV) $\div 2$.
$3x + 4z = 6 + 4z = 26$	(XVII)	$=$ (XIII) with 2 for x.
$z = 5$	(XVIII)	$=$ (XVII) transp. and $\div 4$.
$5x - y + 6z = 10 - y + 30 = 37$	(XIX)	$=$ (VIII) with 2 substituted for x and 5 for z
$y = 3$	(XX)	$=$ (XIX) transposed.
$v + x + y + z = v + 2 + 3 + 5 = 14$ $v = 4$	(XXI)	$=$ (I) with values of x, y, and z substituted.

Therefore the required values are $v = 4$, $x = 2$, $y = 3$, and $z = 5$.

EXERCISE XXXV.

Find the values of the unknown quantities in the following equations:—

1. $\left.\begin{aligned} 2x - 3y + 4z &= 28 \\ 3x + 4y - 5z &= 26 \\ 4x - 5y - 6z &= 16 \end{aligned}\right\}$

2. $\left.\begin{aligned} x + y + z &= 5 \\ 2x - y - 3z &= -5 \\ x + 2y - z &= -1 \end{aligned}\right\}$

3. $\left.\begin{aligned} x + y + z &= 0 \\ 2x + 3y + 4z &= -4 \\ 3x + 6y + 7z &= -6 \end{aligned}\right\}$

4. $\left.\begin{aligned} 3x - 2y - z &= 12 \\ 4x - 3y - 2z &= 17 \\ 5x - 5y - 3z &= 21 \end{aligned}\right\}$

5. $\left.\begin{aligned} x + y + z + v &= 0 \\ 2x - 3y - z - 2v &= 11 \\ x + 2y - 3z + 5v &= -17 \\ 3x + 2y - 4z - v &= -5 \end{aligned}\right\}$

6. $\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{5}{6} \\ \frac{1}{x} + \frac{1}{z} &= \frac{3}{4} \\ \frac{1}{y} + \frac{1}{z} &= \frac{7}{12} \end{aligned}\right\}$

7. $\left.\begin{aligned} x + y &= xy \\ x + z &= 2xz \\ 2(y + z) &= 3yz \end{aligned}\right\}$

8. $\left.\begin{aligned} x + 3y + 2z &= b \\ 3x + 5y - 2z &= m \\ 4x - y + z &= n \end{aligned}\right\}$

9. $\left.\begin{aligned} ax + by &= c \\ bx + cz &= a \\ cy + az &= b \end{aligned}\right\}$

10. $\left.\begin{aligned} v + x + y &= 13 \\ v + x + z &= 17 \\ v + y + z &= 18 \\ x + y + z &= 21 \end{aligned}\right\}$

11. $\left.\begin{aligned} &x + y + z = a + b + c \\ &bx + cy + az = cx + ay + bz = a^2 + b^2 + c^2 \end{aligned}\right\}$

12. $\left.\begin{aligned} x + a(y + z) &= m \\ y + a(x + z) &= n \\ z + a(x + y) &= p \end{aligned}\right\}$

PROBLEMS

PRODUCING SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

Ex. 1. What fraction is that whose numerator being doubled and denominator decreased by unity, the value becomes $\frac{2}{3}$, but the denominator being doubled, and numerator increased by 5, the value becomes $\frac{1}{2}$?

SOLUTION.

Let $\frac{x}{y}$ = the fraction; then x = numerator and y = denominator.

$$\left.\begin{aligned} \frac{2x}{y-1} &= \tfrac{2}{3} \\ \frac{x+5}{2y} &= \tfrac{1}{2} \end{aligned}\right\} \quad \begin{aligned} &(\text{I}). \\ &(\text{II}). \end{aligned}$$

$$\left.\begin{aligned} 6x - 2y &= -2 \\ 2x - 2y &= -10 \end{aligned}\right\} \quad \begin{aligned} &(\text{III}) = (\text{I}) \text{ reduced.} \\ &(\text{IV}) = (\text{II}) \text{ reduced.} \end{aligned}$$

$$4x = 8$$
$$x = 2$$

$$12 - 2y = -2$$
$$-2y = -14$$
$$y = 7$$

Therefore the fraction is $\frac{2}{7}$.

Ex. 2. A certain field is rectangular in form, and its dimensions are such that if it were 4 chains longer and 3 chains wider its area would be 103 chains greater than at present, but if it were 2 chains shorter and 7 chains wider, its area would be 119 chains greater than at present. Required its area.

SOLUTION.

Let x = its length and y = its breadth; hence xy = its present area.

Then $x + 4$ = its length when increased by 4 chains.

$y + 3$ = its breadth when increased by 3 chains.

$(x + 4)(y + 3)$ = its area, which is greater than xy by 103 chains.

Also $x - 2$ = length when decreased by 2 chains.

$y + 7$ = breadth when increased by 7 chains.

Then $(x - 2)(y + 7)$ = its area, which is greater than xy by 119 chains. Hence the two required equations are

$(x+4)(y+3) = xy + 103$	(I)	
$(x-2)(y+7) = xy + 119$	(II)	
$xy + 3x + 4y + 12 = xy + 103$	(III)	= (I) expanded.
$xy + 7x - 2y - 14 = xy + 119$	(IV)	= (II) expanded.
$3x + 4y = 91$	(V)	= (III) transposed and collected.
$7x - 2y = 133$	(VI)	= (IV) transposed and collected.
$14x - 4y = 266$	(VII)	= (VI) × 2.
$17x = 357$	(VIII)	= (V) + (VII).
$x = 21$	(IX)	= (VIII) ÷ 17.
$3x + 4y = 63 + 4y = 91$	(X)	= (V) with 21 substituted for x.
$4y = 28$		
$y + 7$		

Hence the area $= xy = 21 \times 7 = 147$ chains.

Ex. 3. Two plugs are opened in the bottom of a cistern containing 664 gallons of water; after 6 hours one of them becomes stopped, and the cistern is emptied by the other in 20 hours; but had 8 hours elapsed before the stoppage occurred, it would only have required 15h. 36m. more to empty it. Assuming the discharge to be uniform, how many gallons did each plug hole discharge per hour?

SOLUTION.

Let x and y = rates of discharge per hour of the two plug holes.

Then $6x + 6y$ = No. of gals. discharged in 6 hours.

And $20y$ = No. of gals. discharged by second in 20 hours.

Then $6x + 26y = 664$ (I).

Also $8x + 8y$ = No. of gals. discharged in 8 hours by both.

And $15\frac{3}{5}y = \frac{78y}{5}$ = No. of gals. discharged by 2nd in 15h. 36m.

Then $8x + 8y + \dfrac{78y}{5} = 664$	(II)	
$40x + 118y = 3320$	(III)	= (II) reduced.
$120x + 520y = 13280$	(IV)	= (I) × 20.
$120x + 354y = 9960$	(V)	= (III) × 3.

$166y = 3320$	(VI)	$=$ (IV) $-$ (V).
$y = 20$	(VII)	$=$ (VI) $\div 166$.
$6x + 26y = 6x + 520 = 664$	(VIII)	$=$ (I) with 20 substituted for y.
$6x = 144$		
$x = 24$		

Therefore rates of discharge are 24 and 20 gals. per hour.

Exercise XXXVI.

1. Find two numbers such that seven times their sum increased by four times the less is equal to 50, and twice their difference increased by three times the greater is equal to 16.

2. Find two numbers whose sum is equal to a, and such that b times the greater is equal to c times the smaller.

3. Two tons of hay and 35 bushels of oats cost \$44, but if oats were to fall in price 20 per cent. and hay were to rise in price $33\frac{1}{3}$ per cent.. they would cost \$51·20. Required the price of hay and oats.

4. A rectangular garden is of such dimensions that were it 20 yards longer and 24 yards wider it would contain 4180 square yards more than its present area, but if it were 24 yards longer and 20 yards wider, its present area would be increased by only 3860 square yards. Required its present area.

5. Find two numbers such that the sum of one-half of the first and one-third of the second shall be 11; and one-third of the first shall be greater by unity than one-fifth of the second.

6. Divide the number 144 into two parts such that $\frac{4}{7}$ of the greater shall exceed $\frac{7}{9}$ of the less by $1\frac{1}{3}$.

7. Divide the number 48 into two parts such that the greater shall contain 4 as divisor four times as often as it contains the less as divisor.

8. Find three numbers such that the first is equal to $\frac{5}{4}$ of the other two, the second exceeds half the sum of the other two by 6, while the third is less by 3 than $\frac{1}{3}$ of the sum of the first and second.

9. In 4000 lbs. of gunpowder there are 3240 lbs. less of sulphur than of charcoal and saltpetre, and 2760 lbs. less of charcoal

than of sulphur and saltpetre. How many lbs. are there of each?

10. Divide the number 72 into three such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall all be equal to each other?

11. A purse holds 16 shillings and 27 ten cent pieces. Now 11 shillings and 13 ten cent pieces only fill $\frac{9}{16}$ of it. How many will it hold of each?

12. A work is printed so that each page contains a certain number of lines, and each line a certain number of letters. If the page had contained 3 lines more, and each line 4 letters more, then the page would have contained 224 letters more than it now contains, but if there had been 2 lines less on a page and 3 letters less in each line, the page would have contained fewer letters by 145. How many lines are there in a page, and how many letters in a line?

13. A certain number of two digits is such that when divided by 4 less than twice the sum of its digits the quotient is 3, but when divided by 5 more than the difference of its digits the quotient is 13. Required the number, the right hand digit being the greater.

14. A sum of \$81·60 is to be paid in ten cent and twenty-five cent pieces, and $2\frac{1}{2}$ times the number of twenty-five cent pieces exceeds 6 times the number of ten cent pieces by 4. Required the number of each coin.

15. A railway train running from Toronto to Kingston meets with an accident which diminishes its speed by $\frac{1}{a}$th of what it was before, and in consequence of this the train is b hours behind time. If, however, the accident had happened c miles nearer to Kingston, the train would only have been d hours behind time. Required the rate of the train before the accident.

16. A stage set out from Collingwood to Goderich with a certain number of passengers, 4 more being outside than inside. The fare of seven outside passengers is half-a-dollar less than that of 4 inside passengers, and the whole fare received amounted to \$45. At the end of half the journey it took up three more outside and one more inside passenger, in consequence of which the whole fare received was $1\frac{7}{15}$ times what it was before. What was the number of passengers and the fare of each?

17. What number of two digits is that which is equal to twice the product of its digits, or to four times their sum?

18. There is a number of three digits such that the middle digit is the arithmetical mean between the others. If the number be divided by the sum of its digits, the quotient is 48, and if 198 be taken from it, the digits are inverted. Required the number.

19. A given piece of metal which weighs p oz., loses a oz. in water. It is, however, composed of two other metals, A and B, and we know that p oz. of A lose b oz. in water, and p oz. of B lose c oz. in water. How many oz. of each metal are there in the piece?

20. Five gamblers, A, B, C, D, E, throw dice upon condition that he who has the lowest throw shall give all the others the sum which they already have. Each loses in turn, commencing with A, and at the end of the fifth game each has the same sum, $32. How much had each at first?

SECTION VII.

INVOLUTION AND EVOLUTION.

132. Involution is the process of finding any proposed power of a quantity.

133. If the quantity to be involved have a negative sign, then the signs of all the even powers will be positive, and the signs of all the odd powers, negative.

Thus, $(-a)^2 = -a \times -a = +a^2$.

$(-a)^3 = (-a)^2 \times -a = +a^2 \times -a = -a^3$.

$(-a)^4 = (-a)^2 \times (-a)^2 = +a^2 \times +a^2 = +a^4$.

$(-a)^5 = (-a)^4 \times -a = +a^4 \times -a = -a^5$, &c.

134. If the quantity to be involved have a positive sign, then all its powers, both even and odd, will have the positive sign.

NOTE 1.—It follows that no even power of any quantity can be negative, and that all odd powers will have the same sign as the quantity from which they are derived.

NOTE 2.—Since $(a - b)^2 = a^2 - 2ab + b^2$ is a positive quantity, it follows that $a^2 + b^2 > 2ab$, as otherwise $a^2 + b^2 - 2ab$ would be negative. Hence the sum of the squares of any two quantities is greater than twice their product.

135. Since $(a^m)^n = a^m \times a^m \times a^m$........to n factors, it follows (Art. 53) that $(a^m)^n = a^{mn}$, and hence we find a required power of the given power of a quantity by multiplying the exponent of the given power by that of the required power.

136. The Involution of algebraic quantities may be divided into three cases—the involution of monomials, of binomials, and of polynomials.

CASE I.

INVOLUTION OF MONOMIALS.

137. RULE.—*Raise the coefficient to the required power by actual multiplication; also raise the different letters to the required power by multiplying the exponents they already have by the exponent of the required power, and connect the two parts thus obtained so as to form one quantity.*

NOTE.—A fraction is raised to any power by involving both numerator and denominator separately to that power,—a mixed number by involving the equivalent improper fraction.

Ex. 1. $(2a^2xy^3)^4 = 2^4 \times (a^2xy^3)^4 = 16 \times a^8x^4y^{12} = 16a^8x^4y^{12}$.
Ex. 2. $(-3ax^2)^3 = (-3)^3 \times (ax^2)^3 = -27 \times a^3x^6 = -27a^3x^6$.

EXERCISE XXXVII.

Write down the values of—

1. $(2a^2)^3$; $(3ab^3)^2$; $(4m^2)^2$; $(3ab^2c^3)^1$; 1^7; $(2a^2y)^0$; $(3a^2xy^3)$
2. $(-a^3)^4$; $(-2a^2bc^2)^7$; $(-\frac{1}{2}abc^3)^3$; $(-\frac{1}{3}xy^3)^2$; $(-2mx^2y^3)^5$.
3. $(a^2x)^0$; $(-ax^2y^3z^4)^e$; $(3ay^3)^3$; $(-3ay^3)^3$; $(3ay^3)^4$; $(-3ay^3)^4$.

CASE II.

INVOLUTION OF BINOMIALS.

138. By actual multiplication we find that—

$(a+c)^7 = a^7 + 7a^6c + 21a^5c^2 + 35a^4c^3 + 35a^3c^4 + 21a^2c^5 + 7ac^6 + c^7$.

$(a-c)^8 = a^8 - 8a^7c + 28a^6c^2 - 56a^5c^3 + 70a^4c^4 - 56a^3c^5 + 28a^2c^6 - 8ac^7 + c^8$.

We here observe the following facts :—

I. *The first term of the expansion is found by raising the first term of the binomial to the required power.*

II. *The literal part of the second term of the expansion is obtained by prefixing the first term of the expansion with exponent decreased by unity to the simple power of the second term of the binomial.*

III. *In the succeeding terms of the expansion the exponent of the first term of the binomial constantly decreases, while that of the second term of the binomial constantly increases by unity.*

IV. *If we take the coefficient of any term and multiply it by the exponent of the first letter of the same term and divide by the number of the term, the quotient is the coefficient of the next succeeding term.*

V. *When the sign of the binomial is + all the signs of the expansion are +, but when the sign of the binomial is − the signs of the expansion are + and − alternately.*

Ex. 1. $(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$.

Here $\frac{1 \times 5}{1} = 5 =$ coef. of 2nd term; $\frac{5 \times 4}{2} = 10 =$ coef. of 3rd term: $\frac{10 \times 3}{3} = 10 =$ coef. of 4th term, &c.

NOTE.—It will be remarked by the student that in these expansions—

I. The number of terms = one more than the exponent of the required power.

II. The sum of the exponents of each term = the exponent of the required power.

III. When the power is even there is only one middle term, but when the power is odd there are two terms in the middle of the expansion having the same coefficient.

IV. The terms following the middle term have the same coefficients as those preceding it but are reversed in order.

Ex. 2. $(2a-3b)^6 = (2a)^6 - 6(2a)^5(3b) + 15(2a)^4(3b)^2 - 20(2a)^3(3b)^3 + 15(2a)^2(3b)^4 - 6(2a)(3b)^5 + (3b)^6$
$= 64a^6 - 6(32a^5)(3b) + 15(16a^4)(9b^2) - 20(8a^3)(27b^3) + 15(4a^2)(81b^4) - 6(2a)(243b^5) + 729b^6$
$= 64a^6 - 576a^5b + 2160a^4b^2 - 4320a^3b^3 + 4860a^2b^4 - 2916ab^5 + 729b^6$.

Trinomials may be involved by writing them as binomials and proceeding after a manner similar to the above.

Ex. 3. $(a-b-2c)^4 = \{(a-b)-2c\}^4 = (a-b)^4 - 4(a-b)^3(2c) + 6(a-b)^2(2c)^2 - 4(a-b)(2c)^3 + (2c)^4$
$= (a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 - 4(2c)(a^3 - 3a^2b + 3ab^2 - b^3) + 6(4c^2)(a^2 - 2ab + b^2) - 4(8c^3)(a-b) + 16c^4$
$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 - (8a^3c - 24a^2bc + 24ab^2c - 8b^3c) + (24a^2c^2 - 48abc^2 + 24b^2c^2) - (32ac^3 - 32bc^3) + 16c^4$
$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 - 8a^3c + 24a^2bc - 24ab^2c + 8b^3c + 24a^2c^2 - 48abc^2 + 24b^2c^2 - 32ac^3 + 32bc^3 + 16c^4$.

EXERCISE XXXVIII.

Write down the expansions of

1. $(a-b)^9$. 2. $(c+x)^4$. 3. $(x-y)^{10}$. 4. $(a+m)^{11}$.
5. $(2-a)^4$. 6. $(x-3)^5$. 7. $(2a+3)^6$. 8. $(3-2m)^5$.
9. $(3a-2y)^5$. 10. $(2b-5c)^3$. 11. $(3x-4y)^4$. 12. $(ab+3c)^5$.
13. $(2ac-xyz)^3$. 14. $(a+b-c)^3$. 15. $(2a-b-c)^4$.
16. $(2a+2b-3c)^5$. 17. $(1+x-x^2)^4$. 18. $(a-b+2c)^5$.

CASE III.

INVOLUTION OF POLYNOMIALS.

139. No general method can be given for involving polynomials to a given power except by actual multiplication. The *second* power of polynomials, however, may be expeditiously obtained by the following;—

RULE.—*Write down the square of the first term and twice the product of the first term by each succeeding term of the polynomial.*

Under this set down the square of the second term and twice the product of the second term by each succeeding term.

Similarly set down the square of the third term and twice the product of the third term by each succeeding term. And proceed thus through all the terms of the polynomial.

Lastly, add the several results together for the complete square.

$$
\begin{aligned}
\text{Ex. 1.}\quad (a-c-d-f+g-h)^2 = a^2 - 2ac - 2ad - 2af + 2ag - 2ah \\
+ c^2 + 2cd + 2cf - 2cg + 2ch \\
+ d^2 + 2df - 2dg + 2dh \\
+ f^2 - 2fg + 2fh \\
+ g^2 - 2gh \\
+ h^2
\end{aligned}
$$

Here we cannot add the quantities together since they are all unlike.

Ex. 2. $(1 - x + x^2 - \frac{1}{2}x^3 + 2x^4 - \frac{1}{2}x^5)^2$

$$
\begin{array}{lllllllllll}
1 & -\,2x & +\,2x^2 & -\,x^3 & +\,4x^4 & -\,x^5 & & & & & \\
 & & +\,x^2 & -\,2x^3 & +\,x^4 & -\,4x^5 & +\,x^6 & & & & \\
 & & & & +\,x^4 & -\,x^5 & +\,4x^6 & -\,x^7 & & & \\
 & & & & & & +\,\frac{1}{4}x^6 & -\,2x^7 & -\,\frac{1}{2}x^8 & & \\
 & & & & & & & & +\,4x^8 & -\,2x^9 & \\
 & & & & & & & & & & +\,\frac{1}{4}x^{10} \\
\hline
1 & -\,2x & +\,3x^2 & -\,3x^3 & +\,6x^4 & -\,6x^5 & +\,\frac{21}{4}x^6 & -\,3x^7 & +\,\frac{7}{2}x^8 & -\,2x^9 & +\,\frac{1}{4}x^{10}
\end{array}
$$

EXERCISE XXXIX.

1. $(2 + \frac{1}{2}x - 3x^2)^2$. 2. $(x + x^2 - x^3)^2$. 3. $(2x - 3x^2 - \frac{1}{2}x^3)^2$.
4. $(1 - \frac{1}{2}a + 2a^2 - a^3)^2$. 5. $(1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + x^4)^2$.
6. $(2a - ax + 2ax^2)^2$. 7. $(1 + bx - cx^2)^2$.
8. $(a - bx - cx^2 + dx^3)^2$. 9. $(1 - a + b^2x^2 - c^3x^3 + d^4x^4)^2$.
10. $(a + b)^6$. 11. $(a - c)^5$. 12. $(ax - 2)^4$.
13. $(2 - 3x + 4x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4)^2$. 14. $(1 + 2x - x^2 + 2x^3 - x^4)^2$

EVOLUTION.

140. Evolution is the process of finding any required root of a quantity.

141. Since $(+a)^2 = +a^2$ and $(-a)^2$ also $= +a^2$, the square root of a^2 may be either $+a$ or $-a$, and hence we always attach the double sign $\pm$ to the *even roots* of a quantity.

Thus, $\sqrt{x^2y^2} = \pm xy$; $\sqrt[4]{x^8y^{12}} = \pm x^2y^3$; &c.

142. Since all even powers are positive, whether the root be negative or positive, it follows that a negative quantity can have no even root.

NOTE.—Expressions *indicating* an even root of a negative quantity, such as $\sqrt{-a^2}$, $\sqrt{-16m^4}$, $\sqrt[4]{-16a^8}$, $\sqrt[6]{-a^6m^{12}z^{18}}$, &c., are called *imaginary* or *impossible* quantities.

143. The root of a complete odd power has the same sign as the power.

Thus, $\sqrt[3]{-a^3} = -a$; $\sqrt[5]{-32a^{10}b^{25}} = -2a^2b^5$; $\sqrt[3]{27a^6m^3} = 3a^2m$.

CASE I.

EVOLUTION OF MONOMIALS.

144. To extract any root of a monomial:—

RULE.—*Extract the required root of the numerical coefficient, and then extract the root of the literal part by dividing the exponent of each letter by the index of the root to be extracted.*

NOTE 1.—We extract a required root of a fraction by taking the root of the numerator and denominator separately—of a mixed number by taking the root of the equivalent improper fraction.

Ex. $\sqrt[4]{16a^8b^{12}} = \sqrt[4]{16 \times a^2b^3} = 2a^2b^3$; $\sqrt[3]{64a^3b^6} = \sqrt[3]{64 \times ab^2} = 4ab^2$.

NOTE 2.—When the exponent of the literal part is not exactly divisible by the index of the root to be taken, we cannot obtain the root, and consequently we merely indicate its extraction by using the radical sign

and proper index, or by using a fractional exponent. Thus, we cannot find the cube root of a^4 because 4, the exponent of a, is not exactly divisible by 3, the index of the cube root; we therefore represent the root required by the expression $\sqrt[3]{a^4}$ or $a^{\frac{4}{3}}$. Such quantities are called *surds* or *irrational quantities.*

EXERCISE XL.

1. Find the square roots of a^4; x^2y^2; $4a^2m^4$; $64a^2$; $121a^6y^8$.

2. Find the cube roots of $-27a^3$; $64a^6y^9$; $125a^3x^{15}$; $-8a^6y^{12}z^3$.

3. Find the square roots of $\frac{16a^2}{25b^4}$; $-\frac{16a^2}{4m^4}$; $\frac{144x^4y^{18}}{81a^4b^2}$; $\frac{64a^8}{625m^2x^2}$.

4. Find the cube roots of $\frac{64a^{12}y^6}{27m^3}$; $\frac{8a^{21}x^{18}y^{12}}{216b^3c^6}$; $-\frac{343a^3b^9}{64m^6y^{21}}$.

5. Find $\sqrt[4]{\frac{16a^4}{b^{12}}}$; $\sqrt[5]{\frac{32a^{10}x^{20}}{243y^5}}$; $\sqrt[6]{\frac{729m^{12}x^{12}}{64a^{12}}}$; $-\sqrt[7]{\frac{a^{14}m^{21}}{x^{28}y^{42}}}$.

CASE II.

EVOLUTION OF POLYNOMIALS.

SQUARE ROOT.

145. In order to investigate a method for extracting the square root of a polynomial, we take the quantity $a+b$ and square it; this gives us $a^2+2ab+b^2$. Next we seek to find or to devise some process by which we can evolve from this latter quantity its square root, $a+b$. Arranging the square according to the powers of the letter of reference, we readily see that we can get a, the first term of the root, by taking the square root of the first term of the arranged square. Subtracting a^2 we have a remainder $2ab+b^2$. Now we endeavour to find some process by which we may use a, the first term of the root, as a divisor for finding the second term, and knowing that this second term is b, we see at once that we must use $2a$ for a trial divisor, because $2ab \div 2a$ gives b, the second term. Finally, as the divisor multiplied by the last term put in the root, must cancel the remaining part of the dividend, i.e., $2ab+b^2$, we observe that we must add b to the trial divisor in order to complete it.

$$
\begin{array}{rl}
 & a^2+2ab+b^2(a+b \\
 & a^2 \\
\hline
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\
\hline
\end{array}
$$

146. The several steps of the above process give us the following :—

RULE.

I. *Having properly arranged the given square, we take the square root of its first term for the first term of the root, and subtract its square from the given square.*

II. *We double the part of the root already found for a trial divisor.*

III. *We ask how often this trial divisor is contained in the first term of the remainder. This gives us the second term of the root.*

IV. *We place the second term both in the root and also in the trial divisor to complete it.*

V. *We multiply the complete divisor thus obtained by the second term of the root, and subtract.*

VI. *If there be a remainder we again double the part of the root already found, for a new trial divisor; again ask how often the first term of the trial divisor is contained in the first term of the remainder; place the quantity answering this both in the root and in the divisor; multiply the divisor thus completed by the last term put in the root;* and so on.

147. We are led to infer that the above rule will answer in all cases, from observing carefully the law by which any polynomial is raised to the second power, and that the given method for extracting the square root is just the reversal of this process.

Thus, $(a+b)^2 = a^2 + 2ab + b^2$.

$$(a+b+c)^2 = a^2 + 2ab + b^2 + 2(a+b)c + c^2.$$

$$(a+b+c+d)^2 = a^2 + 2ab + b^2 + 2(a+b)c + c^2 + 2(a+b+c)d + d^2,$$

$$(a+b+c+d+e)^2 = a^2 + 2ab + b^2 + 2(a+b)c + c^2 + 2(a+b+c)d + d^2 + 2(a+b+c+d)e + e^2.$$

That is to say:—

The square of any polynomial is equal to the square of the first term, plus twice the product of the first term by the second, plus the

square of the second, plus twice the sum of the first two terms into the third, plus the square of the third term; plus twice the sum of the first three terms into the fourth, plus the square of the fourth term,—and so on.

148. Then also, finding upon trial that the rule holds in every case in which it is tested, we conclude that it is a general rule, and use it as such; and moreover, we derive the arithmetical rule from it.*

Ex. 1. What is the square root of $25a^2 - 30ab + 9b^2$?

OPERATION.

$$\begin{array}{r|l} & 25a^2 - 30ab + 9b^2 \ (\ 5a - 3b = \text{sq. root.} \\ & 25a^2 \\ \hline 10a - 3b \ ) & \quad - 30ab + 9b^2 \\ & \quad - 30ab + 9b^2 \\ \hline \end{array}$$

Ex. 2. What is the square root of $x^4 - 4x^3 + 8x + 4$?

OPERATION.

$$\begin{array}{r|l} & x^4 - 4x^3 + 8x + 4 \ (\ x^2 - 2x - 2 = \text{sq. root.} \\ & x^4 \\ \hline 2x^2 - 2x \ ) & - 4x^3 + 8x + 4 \\ & - 4x^3 + 4x^2 \\ \hline 2x^2 - 4x - 2 \ ) & \quad - 4x^2 + 8x + 4 \\ & \quad - 4x^2 + 8x + 4 \\ \hline \end{array}$$

Ex. 3. What is the square root of $4x^6 + 12x^5 + 5x^4 - 2x^3 + 7x^2 - 2x + 1$?

---

* See Author's National Arithmetic for the investigation of the square root as applied to numbers.

OPERATION.

$$
\begin{array}{lll}
 & 4x^6+12x^5+5x^4-2x^3+7x^2-2x+1 & (\,2x^3+3x^2-x+1 \\
 & 4x^6 & \\
\hline
4x^3+3x^2\,) & \quad 12x^5+5x^4 & \\
 & \quad 12x^5+9x^4 & \\
\hline
4x^3+6x^2-x\,) & \qquad -4x^4-2x^3+7x^2 & \\
 & \qquad -4x^4-6x^3+\ \ x^2 & \\
\hline
4x^3+6x^2-2x+1\,) & \qquad\qquad 4x^3+6x^2-2x+1 & \\
 & \qquad\qquad 4x^3+6x^2-2x+1 & \\
\hline
\end{array}
$$

NOTE 1.—If the given quantity is not an exact square, it is an irrational quantity, and of course its exact square root cannot be extracted.

NOTE 2.—In the above examples, and in all others where an even root is extracted, *all* the terms of the root may have their signs changed, and the resulting expression will still be the root required. (See Art. 141).

EXERCISE XLI.

Extract the square roots of :—

1. $4a^2 + 12ab + 9b^2$; $a^2 - 4ax + 4x^2$; $4a^2x^2 - 28acx + 49c$
2. $9a^2m^2 + 30amxy + 25x^2y^2$; $16a^2x^4 - 8ab^2c^3x^2 + b^4c^6$.
3. $5x^2 + 1 - 6x + 12x^3 + 4x^4$.
4. $x^4 - 2x^2y^2 - 2x^2 + y^4 + 2y^2 + 1$.
5. $a^2 + 2ab - 2ac + b^2 - 2bc + c^2$.
6. $12a^3 + 9a^4 + 34a^2 + 20a + 25$.
7. $a^2 + 2ab + b^2 + 2ac + 2bc + c^2 + 2ad + 2bd + 2cd + d^2$.
8. $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$.
9. $a^4 - 8a^3c + 24a^2c^2 - 32ac^3 + 16c^4$.
10. $1 - 2y + 7y^2 - 2y^3 + 5y^4 + 12y^5 + 4y^6$.
11. $4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4$.
12. $(x - y)^4 - 2(x^2 + y^2)(x - y)^2 + 2(x^4 + y^4)$.
13. $a^4 + b^4 + c^4 + d^4 - 2a^2(b^2 + d^2) - 2b^2(c^2 - d^2) - 2c^2(d^2 - a^2)$.
14. $1 + 2\frac{1}{9}x^2 - \frac{1}{2}x^5 + \frac{1}{16}x^6 - \frac{2}{3}x - \frac{7}{6}x^3 + \frac{7}{6}x^4$.
15. $\left(\dfrac{x}{y}\right)^2 - yx + \frac{1}{4}x^4 - 2 + \dfrac{x^3}{y} + \dfrac{y^2}{x^2}$.

149. THEOREM.—*In the arithmetical extraction of the square root, after* n + 1 *figures of the root have been obtained by the rule,* n *more may be obtained by dividing the last remainder by the last trial divisor.*

DEMONSTRATION.—Let N represent the number whose square root is to be extracted; let a represent the part of the root already found, and let x represent the part of the root yet to be found.

Then $\sqrt{N} = a + x$ $\therefore$ $N = a^2 + 2ax + x^2$.

$N - a^2 =$ the remainder after $n + 1$ figures have been found, and $2a$ is the trial divisor.

Then $\dfrac{N - a^2}{2a} = \dfrac{2ax + x^2}{2a} = x + \dfrac{x^2}{2a}$. If now we can show that $\dfrac{x^2}{2a}$ is a *proper fraction*, we shall show that the integral part of the quotient of the remainder $\div$ the trial divisor, under the given conditions, constitutes the remaining part of the root. By supposition x contains only n digits, therefore x^2 cannot contain more than $2n$ digits, but a by hypothesis consists of the $n + 1$ left hand digits of the root, and must therefore, affixing the n ciphers which are understood, contain $2n + 1$ digits. Hence in the fraction $\dfrac{x^2}{2a}$ the denominator contains $2n + 1$ digits, while the numerator cannot consist of more than $2n$ digits, and therefore $\dfrac{x^2}{2a}$ is a proper fraction, and rejecting it, we get $\dfrac{N - a^2}{2a} = x =$ the remaining digits of the root.

Ex. Find the square root of 12 to 11 places of decimals.

Here we must obtain the first 6 digits by the ordinary rule; this gives us 3·46410 and a rem. 111900, the last trial divisor being 692820. Then 111900 $\div$ 692820 $=$ 16151 $=$ the remaining five digits of the required root, which is therefore $=$ 3·4641016151.

NOTE.—If the given quantity be *not a complete square*, then the approximate square root thus found may possibly differ by a unit of the lowest denomination, from the square root carried out to same number of places by the ordinary rule.

CUBE ROOT.

150. In investigating a method for extracting the cube root of a polynomial, we proceed as follows:—

Taking $a + b$ and cubing it, we get $a^3 + 3a^2b + 3ab^2 + b^3$, and we endeavour to devise some process by which we can evolve from this latter

quantity its known cube root, $a + b$. Having arranged the given cube according to the powers of its letter of reference, we see that we can obtain a, the first term of the root, by taking the cube root of a^3, the first term of the cube. We subtract the cube of a from the whole expression, and bring down the remainder $3a^2b + 3ab^2 + b^3$.

$$\begin{array}{l} a^3 + 3a^2b + 3ab^2 + b^3 \;(\,a + b \\ a^3 \\ \hline 3a^2 + 3ab + b^2)\,3a^2b + 3ab^2 + b^3 \\ \qquad\qquad 3a^2b + 3ab^2 + b^3 \\ \hline \end{array}$$

Next we observe that if we divide the 1st term of this rem. by three times the square of a (the part of root already found), the quotient is b, the required 2nd term of the root. Finally, as all the remainder must be cancelled by the product of the divisor by b, the last term put in the root, we see that we must increase $3a^2$, the trial divisor, by $3ab$ (i.e., three times the product of what was in the root by the term last put in), and b^2 (i. e., the square of the term last put in the root). Upon multiplying the complete divisor $3a^2 + 3ab + b^2$ by b, and subtracting, we find that there is no remainder.

151. The above process enables us to extract the cube root in this particular case, and as it holds good in every case in which it is tested, we conclude that it holds universally. Thus for the extraction of the cube root we get the following :—

RULE.

I. *Arrange the given cube according to some letter of reference.*

II. *Take the cube root of the* 1st *term of the arranged cube, and place it as the* 1st *term of the root.*

III. *Subtract the cube of the* 1st *term of the root from the given cube.*

IV. *Take three times the square of the part of the root already found as a trial divisor.*

V. *Divide the* 1st *term of the remainder by the* 1st *term of the trial divisor, and place the quotient as the* 2nd *term of the root.*

VI. *Complete the trial divisor by adding to it,*

1st, *Three times the product of what was in the root by the term last put in the root; and*

2nd, *The square of the term last put in the root.*

VII. *Multiply the divisor thus completed by the last term put in the root, and subtract the product from the part of the given cube remaining.*

VIII. *Again find a trial divisor, as in* (IV); *divide the* 1*st term of last remainder by the* 1*st term of this trial divisor, and place the quotient as* 3*rd term of the root. Again complete the trial divisor as in* VI, *by making the two additions there described; multiply the complete divisor by the last term put in the root, subtract*,—and so on.

152. We may be led to infer this rule for extracting the cube root of a polynomial by reversing the process by which a polynomial is raised to the third power, as may be seen by an attentive examination of the following:—

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a+b+c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3.$$
$$(a+b+c+d)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3.$$
$$+ 3(a+b+c)^2d + 3(a+b+c)d^2 + d^3.$$

Whence it appears that:—

The cube of any polynomial is equal to the cube of the first term, plus three times the square of the first term multiplied by the second, plus three times the first term multiplied by the square of the second, plus the cube of the second term, plus three times the square of the sum of the first two terms multiplied by the third, plus three times the sum of the first two terms multiplied by the square of the third, plus the cube of the third term, plus three times the square of the sum of the first three terms multiplied by the fourth, plus three times the sum of the first three terms multiplied by the square of the fourth, plus the cube of the fourth term; and so on.

Ex. 1. Find the cube root of $8a^3 - 84a^2x + 294ax^2 - 343x^3$.

OPERATION.

$$\begin{array}{lrrr|l}
 & & & & 8a^3 - 84a^2x + 294ax^2 - 343x^2(2a - 7x \\
 & & & & 8a^3 \\
\hline
3(2a)^2 = & 12a^2 & & & -84a^2x + 294ax^2 - 343x^2 \\
3(2a)(-7x) = & & -42ax & & \\
(-7x)^2 = & & & +49x^2 & -84a^2x + 294ax^2 - 343x^2 \\
\hline
 & 12a^2 & -42ax & +49x^2 &
\end{array}$$

Ex. 2. What is the cube root of $27a^6 - 54a^5 + 63a^4 - 44a^3 + 21a^2 - 6a + 1$?

OPERATION.

$$27a^6 - 54a^5 + 63a^4 - 44a^3 + 21a^2 - 6a + 1 \; (\; 3a^2 - 2a + 1 = \text{root.}$$
$$27a^6$$

1st trial Divisor $= 3(3a^2)^2$	$= 27a^4$	$-54a^5 + 63a^4 - 44a^3 =$ 1st Dividend.
1st Increment $= 3(3a^2) \times (-2a) =$	$-18a^3$	
2nd Increment $= (-2a)^2$	$= +4a^2$	
1st complete Divisor	$= 27a^4 - 18a^3 + 4a^2$	$-54a^5 + 36a^4 - 8a^3 =$ Product of 1st comp. Div. by $-2a$.
2nd trial Div. $= 3(3a^2 - 2a)^2$	$= 27a^4 - 36a^3 + 12a^2$	$27a^4 - 36a^3 + 21a^2 - 6a + 1 =$ 2nd Dividend.
1st Increment $= 3(3a^2 - 2a) \times 1 =$	$9a^2 - 6a$	
2nd Increment $= 1^2$	$= +1$	
2nd complete Divisor	$= 27a^4 - 36a^3 + 21a^2 - 6a + 1$	$27a^4 - 36a^3 + 21a^2 - 6a + 1 =$ 2d comp. Div. $\times + 1$

SAME QUESTION SOLVED BY HORNER'S METHOD.

First Column.	*Second Column.*	
		$27a^6 - 54a^5 + 63a^4 - 44a^3 + 21a^2 - 6a + 1 \; (\; 3a^2 - 2a + 1$
$3a^2$	$9a^4$	$27a^6$
$3a^2$	$18a^4$	
$6a^2$	$27a^4$	$-54a^5 + 63a^4 - 44a^3$
$3a^2$	$-18a^3 + 4a^2$	
$9a^2 - 2a$	$27a^4 - 18a^3 + 4a^2$	$-54a^5 + 36a^4 - 8a^3$
$-2a$	$-18a^3 + 8a^2$	
$9a^2 - 4a$	$27a^4 - 36a^3 + 12a^2$	$27a^4 - 36a^3 + 21a^2 - 6a + 1$
$-2a$	$9a^2 - 6a + 1$	
$9a^2 - 6a + 1$	$27a^4 - 36a^3 + 21a^2 - 6a + 1$	$27a^4 - 36a^3 + 21a^2 - 6a + 1$

EXPLANATION.—The foregoing is a second method of extracting the cube root, known as Horner's method. Upon careful examination it will be seen that the same trial divisors and complete divisors are used as in the other method, but that they are obtained somewhat differently. The several steps are as follows:—

1st. Take the cube root of the first term and place it as first term of the root, also place it to the left of the arranged cube, under the head First Column.

2nd, Multiply the first term of the first column by the first term of the root, and place the product as first term of the second column; also multiply the first term of the second column by the first term of the root, and place it in the third column, i.e., under the given cube, and subtract.

3rd, To the first term of the first column add the first term of the root, multiply the sum by the first term of the root, and place the product as the second term of the second column.

4th, Again add to the first column the first term of the root.

5th, Add the first and second terms of the second column together for a trial divisor. Ascertain how often this goes in the first term of the dividend, and place the quotient $(-2a)$ in the root, and also attach it to the $9a^2$ in the first column.

6th, Multiply the $9a^2 - 2a$ in the first column by $-2a$, the last term put in the root, and place the product $-18a^3 + 4a^2$ under the $27a^4$ in the second column and add; this gives $27a^4 - 18a^3 + 4a^2$ for complete divisor.

7th, Multiply the complete divisor by $-2a$, the term last put in the root, and place the product in the third column.

8th, Subtract and go again through the whole process as before.

EXERCISE XLII.

Extract the cube root of each of the following quantities:—

1. $8x^3 + 36x^2 + 54x + 27$.
2. $a^6 - 40a^3 + 6a^5 + 96a - 64$.
3. $1 - 6a + 12a^2 - 8a^3$.
4. $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$.
5. $8a^3x^3 - 84a^2bx^4 + 294ab^2x^5 - 343b^3x^6$.
6. $8x^6 - 36ax^5 + 102a^2x^4 - 171a^3x^3 + 204a^4x^2 - 144a^5x + 64a^6$.
7. $x^6 - 3x^5 + 6x^4 - 7x^3 + 6x^2 - 3x + 1$.
8. $a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3 + 3(a+b+c)^2d + 3(a+b+c)d^2 + d^3 + 3(a+b+c+d)^2e + 3(a+b+c+d)e^2 + e^3$.

NOTE.—In Ex. 8 endeavour to keep the quantities in brackets, and the labor of extracting the cube root will be materially lightened.

153. THEOREM.—*In the extraction of the cube root of a number when* n + 2 *figures have been found by the ordinary rule,* n *figures more may be found by dividing the remainder by the last trial divisor.*

DEMONSTRATION.—Let N represent the number whose cube root is required; let a represent the $n+2$ figures already found, and let x represent the n remaining figures.

Then $\sqrt[3]{N} = a + x$, $\therefore$ $N = a^3 + 3a^2x + 3ax^2 + x^3$.

$N - a^3 =$ the remainder after $n+2$ figures of the root have been found, and $3a^2 =$ the trial divisor.

$$\frac{N - a^3}{3a^2} = \frac{3a^2x + 3ax^2 + x^3}{3a^2} = x + \frac{x^2}{a} + \frac{x^3}{3a^2}.$$

Now if we can show that $\dfrac{x^2}{a} + \dfrac{x^3}{3a^2}$ is a proper fraction, we shall have proved that, neglecting the remainder arising from the division, we may obtain the next n figures of the root by dividing by the trial divisor. By hypothesis x contains only n digits, while it is manifest that 10^n contains $n+1$ digits; hence $x < 10^n$ and $\therefore x^2 < 10^{2n}$. And since a contains the left hand $n+2$ digits of the root, taking into account the position of these with reference to the decimal point, a must contain $2n+2$ figures. And therefore a is not less than 10^{2n+1}. Hence $\dfrac{x^2}{a} < \dfrac{10^{2n}}{10^{2n+1}}$ that is, $\dfrac{x^2}{a} < \frac{1}{10}$. Similarly $\dfrac{x^3}{3a^2} < \dfrac{10^{3n}}{3 \times 10^{4n+2}}$, that is $< \dfrac{1}{3 \times 10^{n+2}}$.

Hence $\dfrac{x^2}{a} + \dfrac{x^3}{3a^2} < \frac{1}{10} + \dfrac{1}{3 \times 10^{n+2}}$ and $\therefore <$ unity.

Ex. Required the cube root of 10973936869410151220850 48.

Here since there are 26 figures in the cube there are 9 in the root, and we proceed to obtain the first 5 of these by the ordinary rule. The five digits thus obtained are 22222, with a remainder 3291818930151220850 48, and a trial divisor 1481451852 00. Then 329181893015122085048 ÷ 148145185200 = 2222 + = remaining four digits of the root, which is therefore = 222222222.

EXTRACTION OF ROOTS IN GENERAL.

154. By observing the mode of writing the square, cube, &c., of polynomials, we can deduce the following general rule for the extraction of any root of a polynomial:

RULE.

I. *Arrange the given polynomial according to a letter of reference.*

II. *Extract the required root of the first term, this will be the first term of the root.*

III. *Subtract the power of this first term of the root from the given polynomial.*

IV. *Divide the first term of the remainder by twice the first term of the root for the square root, three times its square for the cube root, four times its cube for the fourth root, five times its fourth power for the fifth root*, and so on; *the quotient will be the second term of the root.*

V. *Involve the whole of the root now found to the specified power, and subtract it from the given polynomial.*

VI. *Divide the 1st term of the remainder by the same divisor as before, and the quotient will be the third term of the root. Again involve the whole of the root now found to the specified power; subtract*, and so on.

NOTE.– It is manifest that the rule verifies itself.

Ex. What is the fourth root of $16x^8 - 32x^7 + 88x^6 - 104x^5 + 145x^4 - 104x^3 + 88x^2 - 32x + 16$?

OPERATION.

$$(\text{root} = 2x^2 - x + 2)$$

$$16x^8 - 32x^7 + 88x^6 - 104x^5 + 145x^4 - 104x^3 + 88x^2 - 32x + 16$$

$$(2x^2)^4 = 16x^8$$

$$32x^6) \quad -32x^7 = \text{1st term. of rem.}$$

$$(2x^2 - x)^4 = 16x^8 - 32x^7 + 24x^6 - 8x^5 + x^4.$$

$$32x^6) \quad 64x^6 = \text{1st term. of rem.}$$

$$(2x^2 - x + 2)^4 = 16x^8 - 32x^7 + 88x^6 - 104x^5 + 145x^4 - 104x^3 + 88x^2 - 32x + 16$$

Rem. = 0. Hence $2x^2 - x + 2$ is the fourth root required.

SECTION VIII.

THEORY OF INDICES.

155. It has been stated (Art. 17) that when a fractional index is employed, the numerator of the fraction

denotes the power to be taken, and the denominator indicates the root to be extracted. We have now to add that a *negative exponent* is sometimes employed for the purpose of denoting the *reciprocal of a quantity with the same exponent taken positively.*

Thus, a^{-m} is used to denote $\dfrac{1}{a^m}$ whether m be fractional or integral.

156. THEOREM I. *If* m *and* n *be any positive integral quantities, then* $a^m \times a^n = a^{m+n}$.

DEMONSTRATION. $a^m = a \times a \times a \ldots$ to m factors, and $a^n = a \times a \times a \ldots$ to n factors.

Therefore $a^m \times a^n = a \times a \times a \ldots$ to m factors $\times a \times a \times a \ldots$ to n factors.

$= a \times a \ldots$ to $m + n$ factors $= a^{m+n}$, which was to be proved.

157. THEOREM II. *If* m *and* n *be any positive integral quantities, then* $(a^m)^n = a^{mn} = (a^n)^m$.

DEMONSTRATION. $(a^m)^n = a^m \times a^m \times a^m \ldots\ldots$ to n factors $= a^{m+m+m} \ldots\ldots$ to n terms $= a^{mn}$.

$(a^n)^m = a^n \times a^n \times a^n \ldots$ to m factors $= a^{n+n+n} \ldots$ to m terms $= a^{nm}$.

But $mn = nm$ $\therefore$ $a^{mn} = a^{nm}$, and since $(a^n)^m$ and $(a^m)^n$ are each $= a^{mn}$ $\therefore$ $(a^m)^n = a^{mn} = (a^n)^m$ which was to be proved.

158. THEOREM III. *If* m *and* n *be any positive integers, then the* m*th root of the* n*th power of* a *is equal to the* n*th power of the* m*th root of* a. *That is,* $\sqrt[m]{(a^n)} = (\sqrt[m]{a})^n$.

DEMONSTRATION. Let $\sqrt[m]{(a^n)} = x^n$; raising both to the mth power we get $a^n = (x^n)^m = (x^m)^n$ by the preceding theorem.

Extracting the nth root of each of these we get $a = x^m$; and extracting the mth root of each of these we get $\sqrt[m]{a} = x$; and finally raising each of these to the nth power we have $(\sqrt[m]{a})^n = x^n$. But $\sqrt[m]{(a^n)} = x^n$ $\therefore$ $\sqrt[m]{(a^n)} = (\sqrt[m]{a})^n$, which was to be proved.

159. THEOREM IV. *Both numerator and denominator of a fractional exponent may be multiplied by the same quantity without altering the value of the whole expression, of which it forms part.*

That is, $a^{\frac{m}{n}} = a^{\frac{mr}{nr}}$.

DEMONSTRATION. Let $a^{\frac{m}{n}} = x$. Then $a^m = x^n$; also $a^{mr} = x^{nr}$.

Therefore extracting the nrth root of each, $a^{\frac{mr}{nr}} = x$; but $a^{\frac{m}{n}} = x$.

Therefore $a^{\frac{m}{n}} = a^{\frac{mr}{nr}}$, which was to be proved.

160. THEOREM V. *If* $\frac{m}{n}$ *and* $\frac{r}{s}$ *are any positive fractional quantities, then* $a^{\frac{m}{n}} \times a^{\frac{r}{s}} = a^{\frac{m}{n}+\frac{r}{s}}$.

DEMONSTRATION. By last theorem $a^{\frac{m}{n}} = a^{\frac{ms}{ns}}$ and $a^{\frac{r}{s}} = a^{\frac{nr}{ns}}$.

Therefore $a^{\frac{m}{n}} \times a^{\frac{r}{s}} = a^{\frac{ms}{ns}} \times a^{\frac{nr}{ns}}$

$a^{\frac{ms}{ns}} = (a^{ms})^{\frac{1}{ns}}$ and also $a^{\frac{nr}{ns}} = (a^{nr})^{\frac{1}{ns}}$.

Therefore $a^{\frac{m}{n}} \times a^{\frac{r}{s}} = a^{\frac{ms}{ns}} \times a^{\frac{nr}{ns}} = (a^{ms})^{\frac{1}{ns}} \times (a^{nr})^{\frac{1}{ns}} = (a^{ms} \times a^{nr})^{\frac{1}{ns}}$

$= (a^{ms+nr})^{\frac{1}{ns}} = a^{\frac{ms+nr}{ns}} = a^{\frac{ms}{ns}+\frac{nr}{ns}} = a^{\frac{m}{n}+\frac{r}{s}}$, which was to be proved.

Corollary. Similarly it may be proved that $a^{\frac{m}{n}} \div a^{\frac{r}{s}} = a^{\frac{m}{n}-\frac{r}{s}}$.

161. THEOREM VI. $\left(a^{\frac{m}{n}}\right)^{\frac{r}{s}} = a^{\frac{mr}{ns}}$.

DEMONSTRATION. Let $\left(a^{\frac{m}{n}}\right)^{\frac{r}{s}} = x$, then $\left(a^{\frac{m}{n}}\right)^{r} = x^s$, that is (Art. 157), $a^{\frac{mr}{n}} = x^s$. Therefore $a^{mr} = x^{ns}$, and therefore extracting the nsth root of each, $a^{\frac{mr}{ns}} = x$, but $\left(a^{\frac{m}{n}}\right)^{\frac{r}{s}} = x$ $\therefore$ $\left(a^{\frac{m}{n}}\right)^{\frac{r}{s}} = a^{\frac{mr}{ns}}$, which was to be proved.

162. THEOREM VII. $a^m \times a^n = a^{m+n}$ when m or n, or both m and n are negative quantities.

DEMONSTRATION. First, let either one of the exponents, as for instance n, be a negative quantity.

Then $a^m \times a^n = a^m \times a^{-n} = a^m \times \frac{1}{a^n} = \frac{a^m}{a^n} = a^{m-n} = a^{m+(-n)}$.

Next let both m and n be negative quantities.

Then $a^m \times a^n = a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}} = a^{-m-n}$

$= a^{-m+(-n)}$, which was to be proved.

163. THEOREM VIII. $(a^m)^n = a^{mn}$ when m or n or both m and n are negative quantities.

DEMONSTRATION. First, let n be negative, then $(a^m)^n = (a^m)^{-n}$ $= \frac{1}{(a^m)^n} = \frac{1}{a^{mn}} = a^{-mn} = a^{m \times (-n)}$.

Second, let m be negative.

Then $(a^m)^n = (a^{-m})^n = \left(\frac{1}{a^m}\right)^n = \frac{1}{a^{mn}} = a^{-mn} = a^{-m \times n}$.

Third, let both m and n be negative,

Then $(a^m)^n = (a^{-m})^{-n} = \frac{1}{(a^{-m})^n} = \frac{1}{a^{-mn}}$ (by second part of demonstration) $= a^{mn} = a^{-m \times (-n)}$, which was to be proved.

164. THEOREM IX. $a^n \times b^n = (ab)^n$.

DEMONSTRATION. Let $a^n \times b^n = x$, then $(a^n \times b^n)^{\frac{1}{n}} = x^{\frac{1}{n}}$.
that is, $a \times b = x^{\frac{1}{n}}$ or $ab = x^{\frac{1}{n}}$ $\therefore (ab)^n = x$.
But $a^n \times b^n = x$. Therefore also $a^n \times b^n = (ab)^n$.

Corollary. $(ab)^n = a^n \times b^n$. Similarly $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{(ab)}$, and conversely $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}$.

165. THEOREM X. *Any factor may be transferred from one term of a fraction to the other by changing the sign of its exponent.*

DEMONSTRATION. $\frac{a^m}{b^n} = \frac{a^m}{b^n} \times \frac{b^{-n}}{b^{-n}} = \frac{a^m \times b^{-n}}{b^n \times b^{-n}} = \frac{a^m b^{-n}}{b^{n-n}}$

$= \frac{a^m b^{-n}}{b^0} = \frac{a^m b^{-n}}{1}$.

Again, $\frac{a^m}{b^n} = \frac{a^m}{b^n} \times \frac{a^{-m}}{a^{-m}} = \frac{a^m \times a^{-m}}{b^n \times a^{-m}} = \frac{a^{m-m}}{b^n a^{-m}} = \frac{a^0}{b^n a^{-m}}$

$= \frac{1}{b^n a^{-m}}$, which was to be proved.

166. By these Theorems it has been proved that whether m and n are positive or negative, integral or fractional,

$$a^m \times a^n = a^{m+n}; \; a^m \div a^n = \frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n} = a^m \times a^{-n} = a^{m-n}$$

$$(a^m)^n = a^{mn} = (a^n)^m; \; a^{\frac{m}{n}} = a^{\frac{ms}{ns}}; \; a^n \times b^n = (ab)^n; \; a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$$

$$(ab)^n = a^n \times b^n; \; (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}; \; \frac{a^m}{b^n} = \frac{1}{a^{-m}b^n} = \frac{b^{-n}}{a^{-m}} = a^m b^{-n}$$

That is :—

(I) *Powers of the same quantity are multiplied together by adding their indices.*

(II) *One power of a quantity is divided by another power of the same by subtracting the index of the divisor from that of the dividend.*

(III) *A power of a given power, or a root of a root, is obtained by multiplying together the two indices.*

(IV) *Powers having unlike fractional indices may be reduced to equivalent expressions having fractional indices with a common denominator.*

(V) *A factor may be removed from one term of a fraction to the other by changing the sign of its exponent.*

(VI) *The product of the same root or power of two or more dissimilar quantities is equivalent to the same root or power of their product, and vice versâ.*

ILLUSTRATIVE EXAMPLES.

Ex. 1. $\dfrac{4m}{5\sqrt{a}} = \dfrac{4m}{5a^{\frac{1}{2}}} = \frac{4}{5}ma^{-\frac{1}{2}}$, or $\dfrac{4m}{5\sqrt{a}} = \dfrac{4}{5m^{-1}\sqrt{a}}$.

Ex. 2. $\dfrac{3\sqrt[3]{(ab^2c^4)}}{5\sqrt{(mn^3)}} = \dfrac{3(ab^2c^4)^{\frac{1}{3}}}{5(mn^3)^{\frac{1}{2}}} = \dfrac{3a^{\frac{1}{3}}b^{\frac{2}{3}}c^{\frac{4}{3}}}{5m^{\frac{1}{2}}n^{\frac{3}{2}}} = \frac{3}{5}a^{\frac{1}{3}}b^{\frac{2}{3}}c^{\frac{4}{3}}m^{-\frac{1}{2}}n^{-\frac{3}{2}}$

$= \dfrac{3}{5a^{-\frac{1}{3}}b^{-\frac{2}{3}}c^{-\frac{4}{3}}m^{\frac{1}{2}}n^{\frac{3}{2}}}$.

Ex. 3. $\dfrac{a^{-\frac{3}{4}} b^{-\frac{5}{4}}}{m^{-\frac{1}{2}} c^{-1}} = \dfrac{m^{\frac{1}{2}} c^2}{a^{\frac{3}{4}} b^{\frac{5}{4}}} = \dfrac{(mc^2)^{\frac{1}{2}}}{(a^3b^5)^{\frac{1}{4}}} = \dfrac{\sqrt{(mc^2)}}{\sqrt[4]{(a^3b^5)}}$.

Ex. 4. $a^{-2} \times a^{-3} = a^{-5}$; $a^{-3} \times a^4 = a$; $a^{-7} \times a^3 = a^{-4}$; $a^8 \times a^{-4} = a^4$.

Ex. 5. $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3} + \frac{3}{4}} = a^{\frac{8}{12} + \frac{9}{12}} = a^{\frac{17}{12}}$; $a^{\frac{2}{3}} \times a^{-\frac{2}{5}} = a^{\frac{2}{3} + (-\frac{2}{5})} = a^{\frac{2}{3} - \frac{2}{5}} = a^{\frac{10}{15} - \frac{6}{15}} = a^{\frac{4}{15}}$.

Ex. 6. $a^4 \div a^{-2} = a^{4-(-2)} = a^{4+2} = a^6$; $a^{-3} \div a^{-7} = a^{-3-(-7)} = a^{-3+7} = a^4$.

Ex. 7. $a^{\frac{4}{3}} \div a^{\frac{1}{2}} = a^{\frac{4}{3} - \frac{1}{2}} = a^{\frac{8}{6} - \frac{3}{6}} = a^{\frac{5}{6}}$; $a^{\frac{3}{7}} \div a^{-\frac{3}{4}} = a^{\frac{3}{7} - (-\frac{3}{4})} = a^{\frac{3}{7} + \frac{3}{4}} = a^{\frac{33}{28}}$.

Ex. 8. $(a^2)^3 = a^{2 \times 3} = a^6$; $(a^{-2})^2 = a^{-2 \times 2} = a^{-4}$; $(a^{-3})^{-2} = a^{-3 \times -2} = a^6$; $(a^{\frac{1}{3}})^{-3} = a^{-\frac{1}{3} \times 3} = a^{-1}$; $(a^{-\frac{1}{7}})^{-7} = a$.

Ex. 9. $\{(a^{\frac{2}{3}})^{-\frac{1}{2}}\}^{-6} = (a^{\frac{2}{3} \times -\frac{1}{2}})^{-6} = (a^{-\frac{1}{3}})^{-6} = a^{-\frac{1}{3} \times -6} = a^{\frac{6}{3}} = a^2$.

Ex. 10. $\sqrt[13]{(a^3b^2\sqrt[3]{\{abc^4\sqrt[4]{(\frac{1}{ab^2c^3})}\}})^{12}} = \sqrt[13]{(a^3b^2\{abc^4(a^{-1}b^{-2}c^{-3})^{\frac{1}{4}}\}^{\frac{1}{3}})^{12}}$
$= \left[a^3b^2\left\{abc^4 a^{-\frac{1}{4}}b^{-\frac{2}{4}}c^{-\frac{3}{4}}\right\}^{\frac{1}{3}}\right]^{\frac{12}{13}} = \left(a^3b^2a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{4}{3}}a^{-\frac{1}{12}}b^{-\frac{2}{12}}c^{-\frac{3}{12}}\right)^{\frac{12}{13}}$
$= \left(a^3a^{\frac{1}{3}}a^{-\frac{1}{12}}b^2b^{\frac{1}{3}}b^{-\frac{2}{12}}c^{\frac{4}{3}}c^{-\frac{3}{12}}\right)^{\frac{12}{13}} = \left(a^{3+\frac{1}{3}-\frac{1}{12}}b^{2+\frac{1}{3}-\frac{2}{12}}c^{\frac{4}{3}-\frac{3}{12}}\right)^{\frac{12}{13}}$
$= \left(a^{\frac{39}{12}}b^{\frac{26}{12}}c^{\frac{13}{12}}\right)^{\frac{12}{13}} = (a^{39}b^{26}c^{13})^{\frac{1}{13}} = a^3b^2c.$

Ex. 11. Divide $a^3 - a^{\frac{4}{3}} + 2a^{\frac{1}{3}} - 2 - a^{-\frac{2}{3}} + a^{-3}$ by $a^{\frac{3}{2}} + a^{\frac{2}{3}} - a^{-\frac{1}{3}} - a^{-\frac{3}{2}}$.

OPERATION.

$$\begin{array}{l}
a^{\frac{3}{2}} + a^{\frac{2}{3}} - a^{-\frac{1}{3}} - a^{-\frac{3}{2}}\big) a^3 - a^{\frac{4}{3}} + 2a^{\frac{1}{3}} - 2 - a^{-\frac{2}{3}} + a^{-3} \big(a^{\frac{3}{2}} - a^{\frac{2}{3}} + a^{-\frac{1}{3}} - a^{-\frac{3}{2}} \\
\qquad\qquad\qquad\qquad\quad \underline{a^3 + a^{\frac{13}{6}} - a^{\frac{7}{6}} - 1} \\
\qquad\qquad\qquad\qquad\quad -a^{\frac{13}{6}} - a^{\frac{4}{3}} + a^{\frac{7}{6}} + 2a^{\frac{1}{3}} - 1 - a^{-\frac{2}{3}} + a^{-3} \\
\qquad\qquad\qquad\qquad\quad \underline{-a^{\frac{13}{6}} - a^{\frac{4}{3}} + a^{\frac{1}{3}} + a^{-\frac{5}{6}}} \\
\qquad\qquad\qquad\qquad\qquad\qquad a^{\frac{7}{6}} - a^{-\frac{5}{6}} + a^{\frac{1}{3}} - 1 - a^{-\frac{2}{3}} + a^{-3} \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{a^{\frac{7}{6}} + a^{\frac{1}{3}} - a^{-\frac{2}{3}} - a^{-\frac{11}{6}}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -a^{-\frac{5}{6}} - 1 + a^{-\frac{11}{6}} + a^{-3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -1 - a^{-\frac{5}{6}} + a^{-\frac{11}{6}} + a^{-3}
\end{array}$$

EXERCISE XLIII.

1. Express $\sqrt{a}$; $\sqrt[3]{a^2}$; $\sqrt[4]{a^5}$; $\sqrt{(ab^3c^2)}$; $\sqrt[5]{(abc)^4}$; $\sqrt[5]{(a^2bc^{10})}$ and $\sqrt[n]{(a^mb^rc^s)^t}$ with fractional indices.

2. Express $a^{\frac{1}{3}}$; $b^{\frac{2}{5}}$; $c^{\frac{3}{8}}$; $a^{\frac{1}{2}}b^{\frac{3}{2}}$; $(abc)^{\frac{1}{5}}$; $a^{\frac{3}{7}}b^{\frac{3}{7}}$; $(a^5b^3c)^{\frac{3}{4}}$; $(a^2b^4c^6m^7)^{\frac{2}{5}}$, and $\left(a^{\frac{1}{r}}b^{\frac{2}{n}}c^{\frac{rs}{m}}\right)^{\frac{r}{m}}$ with the radical sign.

3. Express $\frac{2a}{bm}$; $\frac{2}{a}$; $\frac{3a}{m}$; $\frac{m^2}{ac^2}$; $\frac{3abm}{4m^2c^3}$; $\frac{2a^{\frac{1}{3}}m^{\frac{1}{2}}}{5c\sqrt{m}}$; $\frac{3a^{\frac{1}{3}}b\sqrt{(cm^3)}}{a^2b\sqrt[3]{m}}$; $\frac{1}{\sqrt[3]{(ab^2cm^{\frac{1}{4}})}}$, and $\frac{a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}}{m^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}}$ with negative indices, so as to remove all the literal factors into the numerators.

4. Express $2a$; $\frac{b^2}{c}$; $\frac{3am}{\sqrt{c}}$; $\frac{2a}{3x^2y}$; $ab^2c^{\frac{1}{3}}$; $\frac{3axy^2}{2a^3\sqrt{m}}$; $\frac{4ac}{3mx}$ and $\frac{5\sqrt[3]{(mn^2x^4)}}{3\sqrt[2]{(abx^3)}}$, with negative indices so as to remove all the literal factors into the denominators.

5. Express a^{-1}; $2a^2b^{-3}$; $\frac{3a^{\frac{2}{3}}b^{-3}}{m^{-\frac{2}{3}}c^{-\frac{2}{3}}}$; $\frac{b^{-3}}{m^{-3}}$; $\frac{2^{-1}a^{-2}b^{-3}}{3^{-1}c^{-3}m^{-\frac{1}{2}}}$; $\frac{1}{a^{-1}b^{-\frac{1}{2}}c^{-\frac{1}{3}}m^{-\frac{3}{4}}}$; $\left(\frac{a}{b}\right)^{-\frac{1}{4}}$; $\left(\frac{a^{-1}}{b^{-2}}\right)^{-3}$; $\left(\frac{1}{a^{-2}b^{-\frac{1}{2}}c}\right)^{-\frac{1}{3}}$; and $\left\{\left(\frac{a^{-1}}{b^{-2}}\right)^{-n}\right\}^{-m}$ with positive indices.

6. Simplify $\left(a^{-\frac{3}{5}}\times a^{-\frac{2}{3}}\right)^{-3}$ and $\left(a^{\frac{1}{2}}\times a^{-\frac{2}{3}}\times a^{\frac{3}{4}}\right)^{-\frac{7}{3}}$ and $(a^{-3}\times\sqrt{a}\times\sqrt[3]{a}\times\sqrt[4]{a})^{\frac{6}{23}}$.

7. Simplify $\left(\sqrt[3]{\left\{\sqrt{\left(a^{-3}\times\frac{1}{\sqrt{c}}\right)}ac\right\}}\right)^{12}$ and $\left\{\sqrt{(\sqrt{\{\sqrt{a}\}})}\times a^{\frac{1}{4}}\right\}^{-2}$.

8. Simplify the following expression :

$$\left\{\frac{\sqrt[4]{(a^3\sqrt{b})}}{\{(a^3)^{\frac{1}{4}}b^3c^3\}^{-\frac{1}{4}}}\times\frac{a^3b^3c^3}{(\{(a^4)^{\frac{1}{4}}b^6\}^{\frac{1}{2}}c^0)^{-\frac{1}{2}}}\times\{a^{-1}\sqrt[4]{(b)^{-2}}\sqrt[4]{(c^{-21})}\}\right\}^{\frac{16}{3}}$$

9. $$\frac{\{\sqrt[r]{(x^n)}\sqrt[s]{(x^m)}\sqrt[t]{(x^p)}\}^{rst}}{\{\sqrt[s]{(y^n)}\sqrt[q]{(y)^r}\sqrt[4]{(y^7)}\}^{4sq}}\div\left\{\left(\frac{x^{mt}}{y^{4s}}\right)^{r}\right\}^{q}$$

10. Multiply $a^{\frac{3}{2}} - 3ab^{\frac{1}{2}} + 3a^{\frac{1}{2}}b - b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

11. Multiply $a^{\frac{2}{3}} - a^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}$ by $a^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}$.

12. Multiply $4x - 2x^{\frac{1}{2}}y^{-\frac{1}{2}} + 2x^{\frac{1}{2}}z^{\frac{1}{3}} - y^{-1} + y^{-\frac{1}{2}}z^{\frac{1}{3}}$ by $2x^{\frac{1}{2}} + y^{-\frac{1}{2}} - z^{\frac{1}{3}}$.

13. Divide $9x^{-9}y - 4x^{-7}y^{-1}$ by $-3x^{-4}y - 2x^{-3}$.

14. Divide $a + a^{\frac{3}{4}}b^{-\frac{1}{4}} - a^{\frac{1}{4}}b^{-\frac{3}{4}} - b^{-1}$ by $a^{\frac{5}{8}} + a^{\frac{1}{2}}b^{-\frac{1}{8}} + a^{\frac{3}{8}}b^{-\frac{1}{4}} + a^{\frac{1}{4}}b^{-\frac{3}{8}} + a^{\frac{1}{8}}b^{-\frac{1}{2}} + b^{-\frac{5}{8}}$.

15. Divide $x^{-1} + x^{-\frac{1}{3}} - 1 + x^{\frac{1}{3}} + x$ by $x^{-\frac{1}{3}} + x^{\frac{1}{3}} + 1$.

16. Square $a^{\frac{3}{2}} - a + a^{\frac{1}{2}} + 1 - a^{-\frac{1}{2}} - a^{-1} + a^{-\frac{3}{2}}$.

17. Extract the square root of $a^{\frac{2}{3}} + 2a^{\frac{1}{3}} - 1 - 2a^{-\frac{1}{3}} + a^{-\frac{2}{3}}$.

18. Extract the square root of $x^{\frac{4}{3}} - 4x + 10x^{\frac{2}{3}} - 16x^{\frac{1}{3}} + 19 - 16x^{-\frac{1}{3}} + 10x^{-\frac{2}{3}} - 4x^{-1} + x^{-\frac{4}{3}}$.

19. Extract the cube root of $x^{-1}y^3 - 3x^{-\frac{1}{3}}y + 3x^{\frac{1}{3}}y^{-1} - xy^{-3}$.

20. Extract the cube root of $x^2 - 6x^{\frac{5}{3}}y^{\frac{1}{6}} + 21x^{\frac{4}{3}}y^{\frac{1}{3}} - 44xy^{\frac{1}{2}} + 63x^{\frac{2}{3}}y^{\frac{2}{3}} - 54x^{\frac{1}{3}}y^{\frac{5}{6}} + 27y$.

SURDS.

167. A *surd* or an *irrational quantity*, is a quantity which cannot be represented without the aid of a fractional exponent or the radical sign.

Thus, $\sqrt{3}$, $\sqrt{a}$, $\sqrt[3]{2}$, $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$, $\sqrt[3]{(a+b)}$ or $(a+b)^{\frac{1}{3}}$, &c., are surds or irrational quantities.

168. A *rational quantity* is one which does not *necessarily* involve the use of a radical sign or a fractional exponent.

Thus, a, a^2b, $3am$, $(a^2)^{\frac{1}{2}}$, $(8a^3)^{\frac{1}{3}}$, $(32m^5x^{10})^{\frac{1}{5}}$, &c., are rational quantities.

NOTE 1. The last three of the quantities given above are written in the form of surds, but, the power being such that the root indicated in each case can be extracted, the quantities are really rational. Thus, $(a^2)^{\frac{1}{2}} = a$; $(8a^3)^{\frac{1}{3}} = 2a$; $(32m^5x^{10})^{\frac{1}{5}} = 2mx^2$

NOTE 2. The terms rational and irrational are used simply to express the fact that the quantity has or has not some determinable *ratio* to unity. Thus, $\sqrt{2}$ is irrational, because, since it is equal to 1 + a decimal which neither repeats nor terminates, we cannot compare it with unity so as to say that it contains unity, or that unity contains it any definite number of times.

169. Surds are either *entire* or *mixed*. An *entire surd* is one in which the whole expression is affected by the radical sign or fractional index. A *mixed surd* is one composed of two or more factors, one of which is not affected by the radical sign or fractional index.

Thus, $\sqrt{ab}$; $\sqrt{7}$; $(a+b-7c)^{\frac{1}{4}}$; $(ab^2c^3)^{\frac{3}{5}}$ are entire surds.

$2b^{\frac{1}{2}}$; $4\sqrt{5}$; $3(ab)^{\frac{1}{2}}$; $4\sqrt[3]{27}$; $ab(ac^2x^3)^{\frac{1}{4}}$ are mixed surds.

170. In mixed surds the part not affected by the radical sign or fractional index is called the *coefficient of the surd*, and the part affected by the radical sign or fractional index is called the *surd factor*.

171. Surds are either *similar* or *dissimilar*. Similar surds are such as have, or may be made to have, the same surd factor: all others are dissimilar surds.

Thus, $\sqrt{2}$, $7\sqrt{2}$, $(a+b)\sqrt{2}$, $\sqrt{8}$, which is equal to $2\sqrt{2}$, &c., are similar surds. So also $\sqrt[3]{ab}$; $m\sqrt[3]{ab}$; $(a+m)(ab)^{\frac{1}{3}}$, $17x(ab)^{\frac{1}{3}}$, and $pa^{\frac{1}{3}}b^{\frac{1}{3}}$ are similar surds.

172. A surd is said to be reduced to its simplest form when the surd factor is made as small as possible without putting it in the form of a fraction.

NOTE.—A quadratic surd is one in which the fractional index $\frac{1}{2}$ is employed; a cubic surd is one in which the index $\frac{1}{3}$ is employed, &c.

173. To express a rational quantity in the form of a surd:—

RULE.—*Raise it to the power whose root the surd expresses, and place it beneath the radical sign.*

Ex. 1. $2a = (2a)^{\frac{2}{2}} = \left\{(2a)^2\right\}^{\frac{1}{2}} = (4a^2)^{\frac{1}{2}} = \sqrt{(4a^2)}.$

Ex. 2. $a^2m = (a^2m)^{\frac{3}{3}} = (a^6m^3)^{\frac{1}{3}} = \sqrt[3]{(a^6m^3)}.$

174. To reduce a mixed surd to an entire surd :—

RULE.—*Raise the coefficient to the power indicated by the denominator of the surd-index, and place beneath the radical sign the product of this power and the given surd factor.*

Ex. 3. $4\sqrt{2} = \sqrt{16} \times \sqrt{2} = \sqrt{16 \times 2} = \sqrt{32}$; $a\sqrt{m} = \sqrt{a^2}\sqrt{m} = \sqrt{a^2m}.$

Ex. 4. $2\sqrt[3]{7} = \sqrt[3]{8} \times \sqrt[3]{7} = \sqrt[3]{(8 \times 7)} = \sqrt[3]{56}$; $c^2a^{\frac{1}{3}}m^{\frac{1}{3}} = \sqrt[3]{c^6} \times \sqrt[3]{(am)} = \sqrt[3]{(ac^6m)}.$

Ex. 5. $6a\sqrt[3]{\left(\frac{m}{3a}\right)} = \sqrt[3]{(216a^3)} \times \sqrt[3]{\left(\frac{m}{3a}\right)} = \sqrt[3]{\left(216a^3 \times \frac{m}{3a}\right)} = \sqrt[3]{(72a^2m)}.$

175. To reduce an entire surd to a mixed surd :—

RULE.—*Resolve the quantity under the radical sign into two factors, one of which is the greatest possible perfect power of the root indicated. Extract the root of this factor, and place it as coefficient of the remaining surd factor.*

Ex. 6. $\sqrt{72} = \sqrt{36 \times 2} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$; $\sqrt{20a^3} = \sqrt{4a^2 \times 5a} = 2a\sqrt{5a}.$

Ex. 7. $\sqrt[3]{135} = \sqrt[3]{27 \times 5} = \sqrt[3]{27} \times \sqrt[3]{5} = 3\sqrt[3]{5}$; $\sqrt[3]{a^3x^9 - a^6x^3} = \sqrt[3]{a^3x^3(x^6 - a^3)} = ax\sqrt[3]{x^6 - a^3}.$

176. To reduce surds to their simplest form :—

RULE.—*Reduce the entire surd to a mixed surd by last rule, and if the remaining surd factor be fractional, multiply both its numerator and denominator by such a quantity as will enable us to emove the latter from under the radical sign.*

Ex. 8. $\sqrt[3]{432} = \sqrt[3]{216 \times 2} = \sqrt[3]{216} \times \sqrt[3]{2} = 6\sqrt[3]{2}.$

Ex. 9. $\sqrt{\frac{3}{5}} = \sqrt{\frac{3 \times 5}{5 \times 5}} = \sqrt{\frac{15}{25}} = \sqrt{\frac{1}{25} \times 15} = \sqrt{\frac{1}{25}} \times \sqrt{15} = \frac{1}{5}\sqrt{15}.$

Ex. 10. $\frac{5}{2}\sqrt[3]{9\frac{3}{5}} = \frac{5}{2}\sqrt[3]{\frac{48}{5}} = \frac{5}{2}\sqrt[3]{\frac{48 \times 25}{125}} = \frac{5}{2}\sqrt[3]{\frac{8 \times 150}{125}} = \frac{5}{2}\sqrt[3]{\frac{8}{125} \times 150} = \frac{5}{2}\sqrt[3]{\frac{8}{125}} \times \sqrt[3]{150} = \frac{5}{2} \times \frac{2}{5} \times \sqrt[3]{150} = \sqrt[3]{150}.$

177. To compare dissimilar surds so as to determine which is the greater :—

RULE.—*If mixed surds, reduce them to entire surds, then reduce their indices to a common denominator, and raise each surd to the power indicated by the numerator of its surd-index when thus reduced.*

Ex. 11. Compare $3\sqrt[3]{3}$, $4\sqrt{5}$, and $\sqrt[6]{325}$ with one another,

that is, $\sqrt[3]{81}$; $\sqrt{80}$ and $\sqrt[6]{325}$; that is, $(81)^{\frac{1}{3}}$, $(80)^{\frac{1}{2}}$ and $(325)^{\frac{1}{6}}$,

that is, $(81)^{\frac{2}{6}}$, $(80)^{\frac{3}{6}}$ and $(325)^{\frac{1}{6}}$; that is $(81^2)^{\frac{1}{6}}$, $(80^3)^{\frac{1}{6}}$ and $(325)^{\frac{1}{6}}$,

that is, $(6561)^{\frac{1}{6}}$, $(512000)^{\frac{1}{6}}$ and $(325)^{\frac{1}{6}}$, whence it is evident that $4\sqrt{5}$ is the greatest and $\sqrt[6]{325}$ is the least.

178. To add or subtract surds :—

RULE.—*Reduce them to the same surd factor, when similar, and then add or subtract their coefficients. Dissimilar surds are unlike quantities, and we can only indicate their addition or subtraction by connecting them by their proper signs.*

Ex. 12. $4\sqrt{24} + 2\sqrt{54} - \sqrt{6} + 3\sqrt{96} - 5\sqrt{150}$
$= 8\sqrt{6} + 6\sqrt{6} - \sqrt{6} + 12\sqrt{6} - 25\sqrt{6}$
$= (8 + 6 + 12)\sqrt{6} - (1 + 25)\sqrt{6} = 26\sqrt{6} - 26\sqrt{6} = 0\sqrt{6} = 0.$

Ex. 13. $3\sqrt{\frac{2}{5}} - 2\sqrt{\frac{1}{10}} + \sqrt{\frac{5}{2}} = 3\sqrt{\frac{10}{25}} - 2\sqrt{\frac{10}{100}} + \sqrt{\frac{10}{4}}$
$= \frac{3}{5}\sqrt{10} - \frac{1}{5}\sqrt{10} + \frac{1}{2}\sqrt{10} = \frac{2}{5}\sqrt{10} + \frac{1}{2}\sqrt{10} = \frac{9}{10}\sqrt{10}.$

179. To multiply two or more simple surds :—

RULE.—*Reduce them to the same surd index, then multiply the coefficients together for a new coefficient and the surd factors together for a new surd factor.*

Ex. 14. $4\sqrt{7} \times 3\sqrt{14} = 3 \times 4 \times \sqrt{7 \times 14} = 12\sqrt{49 \times 2} = 84\sqrt{2}.$

Ex. 15. $2\sqrt{5} \times 3\sqrt[3]{2} = 2(5)^{\frac{1}{2}} \times 3(2)^{\frac{1}{3}} = 2(5)^{\frac{3}{6}} \times 3(2)^{\frac{2}{6}} = 2\sqrt[6]{125} \times 3\sqrt[6]{4} = 6\sqrt[6]{500}$

180. To divide one simple surd by another:—

RULE.—*Reduce both to the same surd index. Then divide coefficient by coefficient and surd factor by surd factor.*

Ex. 16. $4\sqrt{11} \div 2\sqrt{5} = \frac{4}{2}\sqrt{\frac{11}{5}} = 2\sqrt{\frac{55}{25}} = \frac{2}{5}\sqrt{55}$.

Ex. 17. $(2\sqrt[3]{2} - 3\sqrt[4]{3} + 7\sqrt[6]{5}) \div 5\sqrt{2} = \dfrac{2\sqrt[3]{2}}{5\sqrt{2}} - \dfrac{3\sqrt[4]{3}}{5\sqrt{2}} + \dfrac{7\sqrt[6]{5}}{5\sqrt{2}}$

$= \frac{2}{5}\sqrt[6]{\frac{4}{8}} - \frac{3}{5}\sqrt[4]{\frac{3}{4}} + \frac{7}{5}\sqrt[6]{\frac{5}{8}} = \frac{2}{5}\sqrt[6]{\frac{32}{64}} - \frac{3}{5}\sqrt[4]{\frac{12}{16}} + \frac{7}{5}\sqrt[6]{\frac{40}{64}} = (\frac{2}{5} \times \frac{1}{2}\sqrt[6]{32}) - (\frac{3}{5} \times \frac{1}{2}\sqrt[4]{12}) + (\frac{7}{5} \times \frac{1}{2}\sqrt[6]{40}) = \frac{1}{5}\sqrt[6]{32} - \frac{3}{10}\sqrt[4]{12} + \frac{7}{10}\sqrt[6]{40}$.

181. To find a multiplier which shall rationalize a binomial quadratic surd, and hence to rationalize the denominator of a fraction when it consists of a binomial quadratic surd.

RULE.—*Change the connecting sign of the given binomial quadratic surd, and the resulting expression will be the multiplier required.*

Ex. 18. What multiplier will rationalize $2\sqrt{2} - 3\sqrt{3}$?

Ans. $2\sqrt{2} + 3\sqrt{3}$.

PROOF. $(2\sqrt{2} - 3\sqrt{3}) \times (2\sqrt{2} + 3\sqrt{3}) = 8 - 6\sqrt{6} + 6\sqrt{6} - 27 = 8 - 27 = -19$.

Ex. 19. Rationalize the denominator of the fraction $\dfrac{5\sqrt{2} - \sqrt{7}}{3\sqrt{5} + \sqrt{6}}$.

Here the multiplier is $3\sqrt{5} - \sqrt{6}$.

$$\text{Then } \frac{5\sqrt{2} - \sqrt{7}}{3\sqrt{5} + \sqrt{6}} = \frac{(5\sqrt{2} - \sqrt{7})(3\sqrt{5} - \sqrt{6})}{(3\sqrt{5} + \sqrt{6})(3\sqrt{5} - \sqrt{6})}$$

$$= \frac{15\sqrt{10} - 3\sqrt{35} - 10\sqrt{3} + \sqrt{42}}{45 - 6}.$$

182. To find a multiplier which shall rationalize a trinomial quadratic surd:—

RULE.—*First use as multiplier the given trinomial quadratic surd with one of its connecting signs changed, the result will be a binomial surd which can be rationalized by the last rule.*

Ex. 20 Rationalize the denominator of $\dfrac{1}{\sqrt{5} - \sqrt{2} + 3\sqrt{3}}$.

Here the first multiplier $= \sqrt{5} - \sqrt{2} - 3\sqrt{3}$ or $\sqrt{5} + \sqrt{2} + 3\sqrt{3}$. Use either, say the former.

$$\text{Then } \frac{1}{\sqrt{5} - \sqrt{2} + 3\sqrt{3}} = \frac{\sqrt{5} - \sqrt{2} - 3\sqrt{3}}{\{(\sqrt{5} - \sqrt{2}) + 3\sqrt{3}\}\{(\sqrt{5} - \sqrt{2}) - 3\sqrt{3}\}}$$

$$= \frac{\sqrt{5} - \sqrt{2} - 3\sqrt{3}}{(\sqrt{5} - \sqrt{2})^2 - 27} = \frac{\sqrt{5} - \sqrt{2} - 3\sqrt{3}}{(5 - 2\sqrt{10} + 2) - 27} = \frac{\sqrt{5} - \sqrt{2} - 3\sqrt{3}}{-20 - 2\sqrt{10}}$$

$$= \frac{\sqrt{2} - \sqrt{5} + 3\sqrt{3}}{20 + 2\sqrt{10}}.$$

Next multiply both terms of this by $20 - 2\sqrt{10}$.

$$\text{Then } \frac{\sqrt{2} - \sqrt{5} + 3\sqrt{3}}{20 + 2\sqrt{10}} = \frac{(\sqrt{2} - \sqrt{5} + 3\sqrt{3})(20 - 2\sqrt{10})}{(20 + 2\sqrt{10})(20 - 2\sqrt{10})}$$

$$= \frac{30\sqrt{2} - 24\sqrt{5} + 60\sqrt{3} - 6\sqrt{30}}{400 - 40} = \frac{5\sqrt{2} - 4\sqrt{5} + 10\sqrt{3} - \sqrt{30}}{60}.$$

EXERCISE XLIV.

1. Express $2^{\frac{2}{3}}$; $7^{\frac{3}{2}}$; $2^{\frac{4}{5}}$; $(1\frac{1}{2})^{\frac{2}{3}}$; $(3\frac{1}{4})^{-\frac{2}{5}}$; $3^{\frac{2}{3}}$; $(\sqrt{a^5})^{-\frac{2}{3}}$, as equivalent surds with indices whose numerator is in each case $+1$.

2. Reduce a; 3; $4\frac{1}{2}$; $2a$; $3a^2b$; $4x^2y^3$, to equivalent surds having indices $\frac{1}{2}$, $-\frac{1}{3}$, and $\frac{1}{4}$.

3. Reduce a^2; $\sqrt{3}$; $2a^2b^3$; ac^2; $4\frac{2}{5}$; 3^{-2}; $(1\frac{3}{4})^{-3}$ and $(x^{-1}y^{-2}z^3)^{-1}$ to equivalent surds with indices $-\frac{1}{2}$ and $\frac{1}{3}$.

4. Reduce $4\sqrt{3}$; $5\sqrt{5}$; $2\sqrt{31}$; $4\sqrt{a}$; $\frac{3}{4}(\frac{2}{3})^{\frac{1}{2}}$; and $\frac{1}{2}\left(\frac{a^2}{b}\right)^{-\frac{2}{3}}$ to entire surds.

5. Reduce $\frac{2}{3}\left(\frac{ab}{3}\right)^{\frac{1}{2}}$; $\frac{a}{b}\left(\frac{3}{4}\right)^{\frac{1}{3}}$; $\frac{2}{3}(3\frac{1}{2})^{\frac{1}{2}}$; $\frac{4}{5}(\frac{2}{5})^{\frac{2}{3}}$, and $\frac{3}{4}a(\frac{1}{2}b)^{-\frac{2}{3}}$ to their simplest form.

6. Reduce $3\sqrt[3]{4}$; $2\sqrt[3]{a}$; $3(\frac{2}{3})^{\frac{1}{3}}$; $a(c)^{\frac{1}{4}}$; $2a(\frac{3}{5}a^2)^{-\frac{2}{3}}$; $\frac{2}{5}(\frac{3}{4}m)^{\frac{2}{5}}$ and $(am + pq)\left(\frac{am + pq}{am - pq}\right)^{-\frac{1}{2}}$ to entire surds.

7. Reduce $\sqrt[3]{135}$; $\sqrt{162}$; $\sqrt[4]{80}$; $7\sqrt[3]{324}$; $\frac{1}{2}\sqrt{\frac{3}{7}}$; $\frac{1}{2}\left(\frac{11a}{704m^5}\right)^{-\frac{1}{6}}$ and $(a^3m^6 - a^6m^3 + a^6m^6)^{\frac{1}{3}}$ to their simplest form.

8. Reduce $\sqrt{\left(\frac{ab^2}{6(a+x)}\right)}$; $\frac{a}{b}\sqrt{\left(\frac{c^2m^2}{a^2n}\right)}$; $\sqrt[m]{(a^{m+n}x)}$ and $\sqrt[q]{\left\{\frac{(az-z^2)^{2q}(b+z)}{c+z}\right\}}$ to their simplest forms.

9. Compare as to their magnitude $3\sqrt{2}$ and $3\sqrt[3]{3}$; $3\sqrt[3]{2\frac{1}{8}}$, $2\sqrt{11}$ and $3\sqrt[6]{7}$.

10. Simplify $4\sqrt{18}+3\sqrt{32}-\sqrt{2}-4\sqrt{8}+5\sqrt{98}$; also $8\sqrt{\frac{3}{4}}+\sqrt{60}-\frac{11}{5}\sqrt{15}+\sqrt{\frac{3}{5}}$.

11. Simplify $\sqrt{28}+\sqrt[3]{81}+2\sqrt{63}-2\sqrt[3]{24}$; also $3b^2(a^3c)^{\frac{1}{2}}+\frac{2}{c}(a^5c^3)^{\frac{1}{2}}-c^4\left(\frac{ac}{b^2}\right)^{\frac{1}{2}}$.

12. Simplify $\sqrt[m]{2^m a^{mp+3} b^{mn+5}}+\sqrt[m]{3^m a^{3m-mn+3} b^{m+5}}-\sqrt[m]{a^3b^5c^{2m}}$.

13. Multiply $5\sqrt{6}$ by $3\sqrt{7}$; $3\sqrt{40}$ by $2\sqrt{5}$; $7\sqrt{6}$ by $5\sqrt{10}$; and $3\sqrt{6}$ by $4\sqrt[3]{60}$.

14. Multiply $\sqrt[3]{16}$ by $\sqrt{8}$; $4a^{\frac{1}{2}}$ by $7a^{\frac{2}{3}}$; $2\sqrt{3}$ by $\sqrt[6]{72}$; and $(\sqrt[3]{4}\times 7\sqrt[3]{6})$ by $\frac{1}{2}\sqrt[3]{5}$.

15. Multiply together $\frac{ax}{bc}\sqrt{ax}$, $\frac{by}{cd}\sqrt[3]{by}$ and $\frac{c^2d}{a}\sqrt[4]{cz}$; also $x-\sqrt{xy}+y$ by $\sqrt{x}+\sqrt{y}$.

16. Multiply $4\sqrt{3}+3\sqrt{7}$ by $2\sqrt{2}-4\sqrt{5}$; and $2\sqrt{3}+\frac{2}{3}\sqrt{\frac{3}{8}}$ by $3\sqrt{2\frac{1}{2}}-4\sqrt{3}$.

17. Divide $3\sqrt{2}$ by $4\sqrt{3}$; $5\sqrt{7}$ by $3\sqrt{8}$; $2\sqrt{\frac{2}{3}}$ by $\sqrt{\frac{3}{5}}$; and $2\sqrt{2\frac{1}{2}}$ by $3\sqrt{3\frac{1}{4}}$.

18. Divide $6\sqrt{12}$ by $3\sqrt[3]{7}$; $3\sqrt[3]{4}$ by $2\sqrt{5}$; $4\sqrt{\frac{2}{3}}$ by $3\sqrt[3]{\frac{3}{2}}$ and $4\sqrt[3]{ax}$ by $3\sqrt{ax}$.

19. Divide $\sqrt{2}+3\sqrt{\frac{1}{2}}$ by $\frac{1}{2}\sqrt{\frac{1}{2}}$; $4\sqrt{3}-5\sqrt[3]{4}+6\sqrt[6]{7}$ by $2\sqrt[4]{3}$; and $\sqrt[5]{ab^{n-1}c^2}$ by $\sqrt[5]{\frac{a^3d^{-1}}{c^{n-1}b^{-2}}}$.

20. Rationalize $\sqrt{7}+6$; $\sqrt{3}-\sqrt{2}$; $4\sqrt{3}-6\sqrt{2\frac{1}{2}}$; $\frac{7}{3}\sqrt{\frac{1}{2}}+\frac{3}{5}\sqrt{2}$ and $\frac{2}{3}\sqrt{\frac{1}{5}}-\frac{3}{4}\sqrt{\frac{1}{6}}$.

21. Rationalize the denominators of $\frac{2}{\sqrt{3}+2\sqrt{5}}$; $\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{5}-3\sqrt{6}}$ and $\frac{2\sqrt{3}+\sqrt{11}}{7\sqrt{8}-8\sqrt{7}}$.

22. Rationalize the denominators of $\frac{3}{\sqrt{3}-\sqrt{x}}$; $\frac{a\sqrt{m}-m\sqrt{a}}{a\sqrt{m}+m\sqrt{a}}$ and $\frac{2+3\sqrt{\frac{2}{3}}}{\frac{1}{3}\sqrt{\frac{1}{2}}-\frac{2}{3}\sqrt{\frac{2}{3}}}$.

23. Rationalize the denominator of $\frac{\sqrt{x^2+x+1}-\sqrt{x^2-x-1}}{\sqrt{x^2+x+1}+\sqrt{x^2-x-1}}$.

24. Rationalize the denominators of $\frac{1}{\sqrt{3}-\sqrt{2}+\sqrt{5}}$; $\frac{1-3\sqrt{2}}{1+3\sqrt{2}-\sqrt{3}}$ and $\frac{2+3\sqrt{3}}{1+2\sqrt{3}-\sqrt{2}}$.

THEOREMS.

183. THEOREM I.—*The product of two dissimilar quadratic surds cannot be a rational quantity.*

DEMONSTRATION. Let $\sqrt{a}$ and $\sqrt{b}$ be any two dissimilar surds. Then $\sqrt{a} \times \sqrt{b}$ cannot be equal to r, a rational quantity. For if it be possible let $\sqrt{a} \times \sqrt{b} = r$. Then, squaring, we get $ab = r^2$ $\therefore b = \frac{r^2}{a} = \frac{r^2 a}{a^2} = \frac{r^2}{a^2} a$. Hence extracting the square root we get $\sqrt{b} = \frac{r}{a}\sqrt{a}$; that is, $\sqrt{b}$ may be made to have the same surd factor as $\sqrt{a}$, and therefore $\sqrt{a}$ and $\sqrt{b}$ are similar surds (Art. 171), but by hypothesis they are dissimilar, therefore they are both similar and dissimilar, which is impossible. Hence $\sqrt{a} \times \sqrt{b}$ cannot be equal to a rational quantity.

184. THEOREM II.—*A quadratic surd cannot be equal to the sum or difference of a rational quantity and a quadratic surd.*

DEMONSTRATION. For if it be possible let $\sqrt{a}$, a quadratic surd, be equal to the sum or difference of r, a rational quantity, and $\sqrt{b}$, another quadratic surd, i.e., let $\sqrt{a} = r \pm \sqrt{b}$. Then $a = r^2 \pm 2r\sqrt{b} + b$ $\therefore$ $\pm 2r\sqrt{b} = a - r^2 - b$ or $\pm \sqrt{b} = \frac{a - r^2 - b}{2r}$, that is, a quadratic surd equals a rational quantity, which is impossible from the definition of a surd.

185. THEOREM III.—*A quadratic surd cannot be equal to the sum or difference of two dissimilar quadratic surds.*

DEMONSTRATION. For if it be possible let $\sqrt{a} = \sqrt{b} \pm \sqrt{m}$ where $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{m}$ are dissimilar quadratic surds

Then $a = b \pm 2\sqrt{b} \times \sqrt{m} + m$ $\therefore$ $\mp 2\sqrt{b} \times \sqrt{m} = b + m - a$ or $\sqrt{b} \times \sqrt{m} = \dfrac{b + m - a}{\pm 2}$.

That is, the product of two dissimilar surds equals a rational quantity, which is impossible by Theor. I.

186. THEOREM IV.—*In any equation consisting of rational quantities and quadratic surds the rational parts on each side are equal, and so also are the quadratic surds.*

DEMONSTRATION. Let $a + \sqrt{b} = x + \sqrt{y}$, then $a = x$ and $\sqrt{b} = \sqrt{y}$. For since $a + \sqrt{b} = x + \sqrt{y}$, then $\sqrt{b} = (x - a) + \sqrt{y}$, hence if $x - a$ does not $= 0$, that is, if x does not $= a$ then we have $\sqrt{b} =$ the sum of a rational quantity and a surd, which (Theor. II) is impossible. Therefore $x = a$ and consequently $\sqrt{b} = \sqrt{y}$.

Cor. 1. Hence if $a + \sqrt{b} = x + \sqrt{y}$ then also $a - \sqrt{b} = x - \sqrt{y}$.

Cor. 2. Hence also if $a + \sqrt{b} = 0$, then $a = 0$ and also $\sqrt{b} = 0$, as otherwise we should have $\sqrt{b} = -a$, i. e., a surd $=$ a rational quantity, which is impossible.

187. THEOREM V.—*If the square root of* $a + \sqrt{b} = x + \sqrt{y}$, *then the square root of* $a - \sqrt{b} = x - \sqrt{y}$.

DEMOMSTRATION. Since by hypothesis $\sqrt{(a + \sqrt{b})} = x + \sqrt{y}$, squaring these equals we get $a + \sqrt{b} = x^2 + 2x\sqrt{y} + y$, and $\therefore$ (Theor. IV) $a = x^2 + y$ and $\sqrt{b} = 2x\sqrt{y}$. Then, subtracting equals from equals, we have $a - \sqrt{b} = x^2 - 2x\sqrt{y} + y^2$, whence $\sqrt{(a - \sqrt{b})} = x - \sqrt{y}$.

Cor. Hence if $\sqrt{(\sqrt{a} + \sqrt{b})} = \sqrt{x} + \sqrt{y}$, then also $\sqrt{(\sqrt{a} - \sqrt{b})} = \sqrt{x} - \sqrt{y}$.

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188. Suppose it is required to extract the square root of a binomial, one of whose terms is rational and the other a quadratic surd, we may proceed as follows :—

Let the given binomial whose square root is to be extracted be $9 + 4\sqrt{5}$, and let $\sqrt{x} + \sqrt{y} =$ the required square root.

Then $\sqrt{(9 + 4\sqrt{5})} = \sqrt{x} + \sqrt{y}$ $\therefore$ $9 + 4\sqrt{5} = x + 2\sqrt{xy} + y$.

Hence (Theor. IV) $x + y = 9$, and $2\sqrt{xy} = 4\sqrt{5}$ or $4xy = 80$.

Then $(x + y)^2 = x^2 + 2xy + y^2 = 81$, Subtracting the equals

$4xy$ and 80 from these equals, we get $x^2 - 2xy + y^2 = 1$, whence $x - y = 1$, but $x + y = 9$ $\therefore$ $2x = 10$ and $x = 5$. Also $2y = 8$ and $y = 4$. Hence $\sqrt{x} + \sqrt{y} = \sqrt{5} + \sqrt{4} = \sqrt{5} + 2$ = square root required.

189. Instead, however, of working out the question thus in full, we can easily deduce a general rule for extracting the square root of certain binomials of the kind alluded to.

Thus, let $a + \sqrt{b}$ represent the given binomial, and let $\sqrt{x} + \sqrt{y}$ = the required square root. Thus we have

$\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y}$; then by *Cor.* Theor. v,

$\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}$; multiplying equals by equals we get

$\sqrt{(a^2 - b)} = x - y$; but by squaring the first equation we get

$a + \sqrt{b} = x + 2\sqrt{xy} + y$; therefore by Theor. IV,

$x + y = a$, and we have shown that $x - y = \sqrt{(a^2 - b)}$,

Hence by addition $2x = a + \sqrt{(a^2 - b)}$ $\therefore$ $x = \frac{1}{2}\{a + \sqrt{(a^2 - b)}\}$,

By subtraction $2y = a - \sqrt{(a^2 - b)}$ $\therefore$ $y = \frac{1}{2}\{a - \sqrt{(a^2 - b)}\}$,

And substituting these values for x and y in the first equation we get the square root required.

Ex. 1. Find the square root of $11 + 6\sqrt{2}$.

OPERATION.

Let $\sqrt{11 + 6\sqrt{2}} = \sqrt{x} + \sqrt{y}$	(I)	
Then $\sqrt{11 - 6\sqrt{2}} = \sqrt{x} - \sqrt{y}$	(II)	Theor. v *Cor.*
$\sqrt{121 - 72} = x - y$	(III)	= (I) × (II).
$\sqrt{49} = x - y$	(IV)	= (III) reduced.
$\therefore x - y = 7$	(V)	
$11 + 6\sqrt{2} = x + 2\sqrt{xy} + y$	(VI)	= (I) squared.
$\therefore x + y = 11$	(VII)	from (VI) by Theor. IV
But $x - y = 7$	(V)	
$\therefore 2x = 18$ and $x = 9$	(VIII)	= (VII) + (V).
Also $2y = 4$ and $y = 2$	(IX)	= (VII) − (V).

Hence $\sqrt{11 + 6\sqrt{2}} = \sqrt{x} + \sqrt{y} = \sqrt{9} + \sqrt{2} = 3 + \sqrt{2}$.

EXERCISE XLV.

Find the square roots of :—

1. $6+\sqrt{20}$.
2. $12-\sqrt{140}$.
3. $32+\sqrt{63}$.
4. $23-2\sqrt{22}$.
5. $10-\sqrt{96}$.
6. $42+3\sqrt{174\cdot2}$.
7. $2+\sqrt{3}$.
8. $43-15\sqrt{8}$.
9. $a-2\sqrt{a-1}$.
10. $2a+2\sqrt{a^2-b^2}$.
11. $8+\sqrt{39}$.
12. $\frac{a^2}{4}+\frac{1}{2}b\sqrt{a^2-b^2}$.

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190. It appears from Art. 189 that when a^2-b is not a perfect square, $\sqrt{x}$ and $\sqrt{y}$ will be complex surds, and the expression $\sqrt{x}+\sqrt{y}$ will be more complex than the given expression $\sqrt{(a+\sqrt{b})}$. Sometimes, however, the square root may be similarly found of a binomial consisting of the sum or difference of two quadratic surds, i.e., a binominal *both* of whose terms are quadratic surds. This is evident from the fact that $\sqrt{a^2c}\pm\sqrt{b}$ may be written $\sqrt{c}\,(a\pm\sqrt{\frac{b}{c}})$, and then, as above, if $a^2-\frac{b}{c}$ be a perfect square, the square root of $a\pm\sqrt{\frac{b}{c}}$ may be represented by $\sqrt{x}\pm\sqrt{y}$.

Ex. Extract the square root of $\sqrt{27}+2\sqrt{6}$.

OPERATION.

$\sqrt{27}+2\sqrt{6}=\sqrt{9}\sqrt{3}+2\sqrt{2}\sqrt{3}=\sqrt{3}(\sqrt{9}+2\sqrt{2})=\sqrt{3}(3+2\sqrt{2})$.

Hence $\sqrt{(\sqrt{27}+2\sqrt{6})}=\sqrt{\{\sqrt{3}(3+2\sqrt{2})\}}=\sqrt[4]{3}\sqrt{3+2\sqrt{2}}$.

Let $\sqrt{3+2\sqrt{2}}=\sqrt{x}+\sqrt{y}$, then $\sqrt{3-2\sqrt{2}}=\sqrt{x}-\sqrt{y}$.

And $\sqrt{9-8}=x-y \therefore x-y=1$.

But $3+2\sqrt{2}=x+2\sqrt{xy}+y \therefore x+y=3$.

Hence $2x=4$ and $x=2$; $2y=2$ and $y=1$.

Therefore $\sqrt{3+2\sqrt{2}}=\sqrt{2}+1$, and $\sqrt[4]{3}(\sqrt{2}+1)=\sqrt[4]{3}(\sqrt[4]{4}+\sqrt[4]{1})$ $=\sqrt[4]{12}+\sqrt[4]{3}$.

EXERCISE XLVI.

Find the square roots of :—

1. $\sqrt{32}-\sqrt{24}$.
2. $3\sqrt{5}+\sqrt{40}$.
3. $3\sqrt{6}+2\sqrt{12}$.
4. $\sqrt{18}-4$.

IMAGINARY QUANTITIES.

191. An imaginary quantity is an expression which represents an even root of a negative quantity. (See Art. 142).

Thus, $\sqrt{-1}$; $\sqrt{-a}$; $\sqrt[4]{-1}$; $\sqrt[n]{-a}$; $\sqrt[6]{-a}$, &c., are imaginary quantities. We can *approximate* to the value of surd quantities, but we cannot even indicate an approximation to the value of an imaginary quantity, which must therefore be regarded as a purely symbolical expression. Such expressions, however, often occur in practice, and so far from being useless have lent their aid in the solution of questions requiring the most skillful and delicate analysis.

192. Imaginary quantities may be added, subtracted, multiplied, divided, &c., like ordinary surds, attention being paid to the few simple principles given in next paragraph.

193. I. Any imaginary quantity may be reduced so as to involve only the imaginary expression $\sqrt{-1}$; because $\sqrt{-a^2}$ $= \sqrt{a^2 \times -1} = \sqrt{a^2}\sqrt{-1} = \pm\, a\sqrt{-1}$. So also $\sqrt{-a} = \sqrt{a}\sqrt{-1}$;

II. $(\sqrt{-a})^2 = -a$, that is $\sqrt{-a} \times \sqrt{-a} = -a$. For though it is true that $\sqrt{-a} \times \sqrt{-a} = \sqrt{-a \times -a} = \sqrt{a^2} = \pm\, a$, we say here that $\sqrt{a^2} = -a$ because we know that the a^2 has arisen from squaring $-a$. We only use the double sign $\pm$ where we wish to indicate that a^2 *might* have arisen from squaring either $+a$ or $-a$.

III. $(\sqrt{-1})^1 = \sqrt{-1}$; $(\sqrt{-1})^2 = -1$; $(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = -1 \times \sqrt{-1} = -\sqrt{-1}$; $(\sqrt{-1})^4 = \{(\sqrt{-1})^2\}^2 = (-1)^2 = +1$, and, since every whole number may be expressed by one of the four expressions $4n$, $4n+1$, $4n+2$, $4n+3$, according as when divided by 4 it leaves a remainder of 0, 1, 2, or 3, and $(\sqrt{-1})^{4n+1} = \sqrt{-1}$; $(\sqrt{-1})^{4n+2}$

$= -1$; $(\sqrt{-1})^{4n+3} = -\sqrt{-1}$ and $(\sqrt{-1})^{4n} = +1$, it follows that the formulæ $\sqrt{-1}$, -1, $-\sqrt{-1}$, and $+1$ express all the powers of $\sqrt{-1}$.

IV. $\sqrt{-a} \times \sqrt{-b} = \sqrt{a}\sqrt{-1} \times \sqrt{b}\sqrt{-1} = \sqrt{ab}(\sqrt{-1})^2 = \sqrt{ab} \times -1 = -\sqrt{ab}$.

Ex. 1. The sum of $\sqrt{-8} + \sqrt{-18} = \sqrt{4}\sqrt{-2} + \sqrt{9}\sqrt{-2} = 2\sqrt{-2} + 3\sqrt{-2} = 5\sqrt{-2}$.

Ex. 2. The sum of $3 - \sqrt{-64} - (2 + \sqrt{-1}) = 3 - \sqrt{64}\sqrt{-1} - 2 - \sqrt{-1} = 3 - 8\sqrt{-1} - 2 - \sqrt{-1} = 1 - 9\sqrt{-1}$.

Ex. 3. $(2\sqrt{-2})(3\sqrt{-3}) = (2\sqrt{2}\sqrt{-1})(3\sqrt{3}\sqrt{-1}) = 6\sqrt{6}(\sqrt{-1})^2 = (6\sqrt{6}) \times -1 = -6\sqrt{6}$.

Ex. 4. $(1 + \sqrt{-1})^2 = 1 + 2\sqrt{-1} + (\sqrt{-1})^2 = 1 + 2\sqrt{-1} - 1 = 2\sqrt{-1}$.

Ex. 5. $(5 + \sqrt{-7})(5 - \sqrt{-7}) = (5)^2 - (\sqrt{-7})^2 = 25 - (-7) = 25 + 7 = 32$.

Ex. 6. $2\sqrt{8} - \sqrt{-10} \div -\sqrt{-2} = \dfrac{2\sqrt{8}}{-\sqrt{-2}} - \dfrac{\sqrt{-10}}{-\sqrt{-2}} = \dfrac{2\sqrt{4}\sqrt{2}}{-\sqrt{-2}} - \dfrac{\sqrt{5}\sqrt{-2}}{-\sqrt{-2}} = \dfrac{4\sqrt{2}}{-\sqrt{-2}} - \dfrac{\sqrt{5}\sqrt{-2}}{-\sqrt{-2}} = \dfrac{-\sqrt{5}\sqrt{-2}}{-\sqrt{-2}} + \dfrac{4\sqrt{-2}\sqrt{-1}}{-\sqrt{-2}}$
$= \dfrac{\sqrt{5}(-\sqrt{-2})}{-\sqrt{-2}} + (4\sqrt{-1})\dfrac{\sqrt{-2}}{-\sqrt{-2}} = \sqrt{5} + 4\sqrt{-1}(-1) = \sqrt{5} - 4\sqrt{-1}$.

Ex. 7. Find the square root $2 + 4\sqrt{-42}$.

Let $\sqrt{2 + 4\sqrt{-42}} = \sqrt{x} + \sqrt{y}$.

$$\sqrt{2 - 4\sqrt{-42}} = \sqrt{x} - \sqrt{y}.$$

$$\sqrt{4 - 16 \times -42} = \sqrt{4 + 672} = \sqrt{676} = 26 = x - y.$$

Also $2 + 4\sqrt{-42} = x + y + 2\sqrt{xy}$ $\quad\therefore\quad 2 = x + y$.

Hence $x = 14$ and $y = -12$ and $\sqrt{x} + \sqrt{y} = \sqrt{14} + \sqrt{-12} = \sqrt{14} + 2\sqrt{-3}$.

EXERCISE XLVII.

Find the value of :—

1. $(4\sqrt{-27}) - (2\sqrt{-12})$ and also of $(a + \sqrt{-b}) + (a + \sqrt{-c})$.
2. The sum of $\sqrt{-5}$, $\sqrt{-7}$ and $\sqrt{-11}$.

3. The square root of $7 + 6\sqrt{-2}$.
4. $(4\sqrt{-3} + 7\sqrt{-2}) \times (4\sqrt{-3} - 7\sqrt{-2})$.
5. The square of $(\sqrt{-2} - 3\sqrt{-3})$.
6. $\dfrac{1}{\sqrt{2} + \sqrt{-5}}$ with denominator rationalized.
7. $(a\sqrt{-1})^{123}$; $(\sqrt{-1})^{72}$; $(\sqrt{-1})^{77}$, and $(\sqrt{-1})^{26}$.
8. The square of $(a - \sqrt{-a})$.
9. The cube of $\sqrt{2} - \sqrt{-4}$.
10. The square root of $-2 - 2\sqrt{-15}$.
11. The square root of $\sqrt{-1}$ and of $-\sqrt{-1}$.
12. The square root of $31 + 42\sqrt{-2}$.
13. $(4 + \sqrt{-2})$ divided by $(2 - \sqrt{-2})$.
14. $14 - \sqrt{15} - (7\sqrt{3} + 2\sqrt{5})\sqrt{-1}$ divided by $7 - \sqrt{-5}$.
15. $(a + b\sqrt{-1})$ multiplied by $(a - b\sqrt{-1})$.*

SECTION IX.

QUADRATIC EQUATIONS.

194. A quadratic equation is one which involves the *second power* of the unknown quantity, but no higher power than the second.

NOTE.—Quadratic equations, like equations of the first degree, may involve only one unknown quantity, or they may involve two or more unknown quantities. In the latter case they are called *simultaneous quadratic equations.*

195. Quadratic equations are of two kinds:—

I. Pure Quadratic Equations; and

II. Affected Quadratic Equations.

196. A *Pure Quadratic Equation* is one which involves, when reduced, only the second power of the unknown quantity.

* This example indicates a mode of resolving $a^2 + b^2$ into factors.

Thus, $x^2 = a$; $x^2 = 9$; $x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 = 16$; $x^{\frac{10}{25}} = x^{\frac{2}{5}} = (x^{\frac{1}{5}})^2 = 4$; $ax^2 + b = cx^2 - m$, &c., are pure quadratics.

197. An *Adfected Quadratic Equation* is one which involves the *first* power as well as the *second* power of the unknown quantity.

Thus, $x^2 + 6x = 27$; $ax^2 - bx = c$, $4x^2 - 3x = 2x - x^2 + a$, &c., are adfected quadratic equations.

198. Any equation may be solved as a quadratic if, when reduced by transposition, &c., the unknown quantity appears in but two terms and its exponent in one, is double that in the other. Thus $x^{\frac{1}{8}} + x^{\frac{1}{16}} = 3$, $x - 5\sqrt{x} = 50$; $\sqrt{x} + 3\sqrt[4]{x} = 9$, $x^4 - 2x^2 = 8$, &c., may be solved as quadratics, but they are not properly speaking quadratic equations.

199. Equations involving surds are generally capable of being solved only by the methods employed for quadratic equations, but they are frequently reducible to simple equations by the following:—

RULE.—*Arrange the surd terms on one or both sides of the equation, as appears most convenient; square both sides of the equation, transpose and reduce; again square if necessary, and so on.*

Ex. 1. Given $\sqrt{7 + \sqrt{6 + \sqrt{x}}} = 3$ to find the value of x.

OPERATION.

$\sqrt{7 + \sqrt{6 + \sqrt{x}}} = 3$	(I)	
$7 + \sqrt{6 + \sqrt{x}} = 9$	(II)	= (I) squared.
$\sqrt{6 + \sqrt{x}} = 2$	(III)	= (II) transposed and reduced.
$6 + \sqrt{x} = 4$	(IV)	= (III) squared.
$\sqrt{x} = -2$	(V)	= (IV) transposed and reduced.
$x = 4$	(VI)	= (V) squared.

Ex. 2. Given $\sqrt{\{x + 2\sqrt{(ax+a^2)}\}} - \sqrt{x} = \sqrt{a}$ to find the value of x.

OPERATION.

$\sqrt{\{x+2\sqrt{(ax+a^2)}\}} - \sqrt{x} = \sqrt{a}$	(I)	
$\sqrt{\{x+2\sqrt{(ax+a^2)}\}} = \sqrt{a} + \sqrt{x}$	(II)	= (I) transposed.
$x + 2\sqrt{(ax+a^2)} = a + 2\sqrt{ax} + x$	(III)	= (II) squared.
$2\sqrt{(ax+a^2)} = a + 2\sqrt{ax}$	(IV)	= (III) transposed.
$4ax + 4a^2 = a^2 + 4a\sqrt{ax} + 4ax$	(V)	= (IV) squared.
$4\sqrt{ax} = 3a$	(VI)	= (V) transp. and then $\div a$.
$16ax = 9a^2$	(VII)	= (VI) squared.
$x = \frac{9}{16}a$	(VIII)	= (VII) $\div a$ and then $\div 16$.

Ex. 3. Given $\sqrt[m]{a+x} = \sqrt[2m]{x^2+5ax+b^2}$ to find the value of x.

OPERATION.

$\sqrt[m]{a+x} = \sqrt[2m]{x^2+5ax+b^2}$	(I)	
$a + x = \sqrt[2]{x^2+5ax+b^2}$	(II)	= (I) raised to the m^{th} power.
$a^2 + 2ax + x^2 = x^2 + 5ax + b^2$	(III)	= (II) squared.
$3ax = a^2 - b^2$	(IV)	= (III) transp. and reduced.
$x = \dfrac{a^2-b^2}{3a}$	(V)	= (IV) $\div 3a$.

Ex. 4. Given $\dfrac{\sqrt{9x}-4}{\sqrt{x}+2} = \dfrac{15+\sqrt{9x}}{\sqrt{x}+40}$ to find the value of x.

OPERATION.

$\dfrac{\sqrt{9x}-4}{\sqrt{x}+2} = \dfrac{15+\sqrt{9x}}{\sqrt{x}+40}$	(I)	
$\left.\begin{array}{l}3x - 4\sqrt{x} + 40\sqrt{9x} - 160 \\ \quad = 15\sqrt{x} + 3x + 30 + 2\sqrt{9x}\end{array}\right\}$	(II)	= (I) cleared of fractions.
$-4\sqrt{x}+120\sqrt{x}-15\sqrt{x}-6\sqrt{x} = 30+160$	(III)	= (II) transp. and red.
$95\sqrt{x} = 190$	(IV)	= (III) collected.
$\sqrt{x} = 2$	(V)	= (IV) $\div 95$.
$x = 4$	(VI)	= (V) squared.

EXERCISE XLVIII.

Find the value of x in the following equations :—

1. $\sqrt{12+x} = 2 + \sqrt{x}$.

2. $\dfrac{x-2}{\sqrt{x}} = \dfrac{2\sqrt{x}}{3}$.

3. $\sqrt{x-24} = \sqrt{x} - 2$.

4. $\sqrt{x} - \sqrt{ax^{-1}} = \sqrt{a+x}$.

5. $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{x+123}+4}+5}+6}+7} = 2$.

6. $\sqrt{a} + \sqrt{x} = \sqrt{ax}$.

7. $\sqrt{2x + \sqrt{x^4 - x^2}} = x + 2$.

8. $\dfrac{\sqrt{x}+28}{4+\sqrt{x}} = \dfrac{38+\sqrt{x}}{\sqrt{x}+6}$.

9. $\sqrt{x} + \sqrt{x+2} = 4(2+x)^{-\frac{1}{2}}$

10. $\sqrt{a+x} + \sqrt{a-x} = \sqrt{ax}$.

11. $a + x = \sqrt{a^2 + x\sqrt{b^2+x^2}}$.

12. $b + x + \sqrt{(b^2 + ax + x^2)} = a$.

13. $\dfrac{\sqrt{x}+2a}{b+\sqrt{x}} = \dfrac{4a+\sqrt{x}}{\sqrt{x}+3b}$.

14. $\sqrt{x} + \sqrt{4a+x} = 2a\,(1+x)^{-\frac{1}{2}}$.

15. $\sqrt{x-32} = 16 - \sqrt{x}$.

16. $\left(\dfrac{b}{a+x}\right)^{\frac{1}{2}} + \left(\dfrac{c}{a-x}\right)^{\frac{1}{2}} = \left(\dfrac{4bc}{a^2-x^2}\right)^{\frac{1}{4}}$

17. $\sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}} = \dfrac{3}{2}\left(\dfrac{x}{x+\sqrt{x}}\right)^{\frac{1}{2}}$

18. $\sqrt{x+a} = c - \sqrt{x+b}$.

19. $x^{-1} + a^{-1} = \sqrt{a^{-2} + \sqrt{4a^{-2}x^{-2} + 9x^{-4}}}$

20. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}} = m$.

---

200. To solve pure quadratics we proceed by the following :—

RULE.—*Having reduced the equation to the form of $x^2 = a$, extract the square root of each side, and prefix the double sign $\pm$ to the right-hand member of the resulting equation.*

Ex. 1. Given $x^2 = a$ to find the values of x.

OPERATION.

$x^2 = a^2$	(I)	
$x = \pm a$	(II)	$= 1$ with square root extracted

NOTE.—The young student in Algebra is sometimes at a loss to know why the double sign $\pm$ is not also prefixed to the left-hand member, since extracting the square root of each side does really give $\pm x = \pm a$ instead of $x = \pm a$. The former of these expressions is, however, easily reducible to the latter. Thus, if $\pm x = \pm a$, then $+ x = + a$, or $+ x = - a$, or $- x = + a$, or $- x = - a$, but the last two of these expressions are equivalent to the first two transposed. So that on the whole $x = a$ or $x = - a$, that is, $x = \pm a$. It appears from this that when we extract the square root of the two members of an equation it is sufficient to put the double sign before the root of one of the members.

Ex. 2. Given $4x^2 + 11 = x^2 + 14$, to find the values of x.

OPERATION.

$4x^2 + 11 = x^2 + 14$	(I)	
$3x^2 = 3$	(II)	= (I) transposed and collected.
$x^2 = 1$	(III)	= (II) $\div$ 3.
$x = \pm 1$	(IV)	= (III) with $\sqrt{}$ of each member taken.

Ex. 3. Given $3x^2 - 4 = \dfrac{x^2 + 2}{5x^0}$ to find the values of x.

OPERATION.

$x^2 - 4 = \dfrac{x^2 + 2}{5x^0}$	(I)	
$15x^2 - 20 = x^2 + 2$	(II)	= (I) $\times 5x^0$, i. e. $\times$ 5 since $x^0 = 1$.
$14x^2 = 22$	(III)	= (II) transposed.
$x^2 = \frac{11}{7}$	(IV)	= (III) $\div$ 14

$$x = \pm\sqrt{\tfrac{11}{7}} = \pm\sqrt{\tfrac{77}{49}} = \pm\sqrt{\tfrac{1}{49} \times 77} = \pm\tfrac{1}{7}\sqrt{77}$$

Ex. 4. Given $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}$ to find the values of x.

OPERATION.

$x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}$	(I)	
$x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2$	(II)	= (I) $\times \sqrt{a^2 + x^2}$
$x\sqrt{a^2 + x^2} = a^2 - x^2$	(III)	= (II) transposed.
$a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4$	(IV)	= (III) squared.
$3a^2x^2 = a^4$	(V)	= (IV) transposed
$x^2 = \dfrac{a^2}{3}$	(VI)	= (V) $\div 3a^2$.

$$x = \pm a\sqrt{\tfrac{1}{3}} = \pm a\sqrt{\tfrac{3}{9}} = \pm\tfrac{1}{3}a\sqrt{3}.$$

Ex. 5. Given $\dfrac{\sqrt{a^2 - x^2} - \sqrt{c^2 + x^2}}{\sqrt{a^2 - x^2} - \sqrt{c^2 + x^2}} = \dfrac{b}{d}$ to find the value of x.

OPERATION.

$$
\begin{array}{l|c|l}
\dfrac{\sqrt{a^2 - x^2} - \sqrt{c^2 + x^2}}{\sqrt{a^2 - x^2} - \sqrt{c^2 + x^2}} = \dfrac{b}{d} & (\text{I}) & \\
\dfrac{2\sqrt{a^2 - x^2}}{-2\sqrt{c^2 + x^2}} = \dfrac{b + d}{b - d} & (\text{II}) & = (\text{I}) \text{ taken as in Art. 106 (VII).} \\
\dfrac{a^2 - x^2}{c^2 + x^2} = \dfrac{(b + d)^2}{(b - d)^2} & (\text{III}) & = (\text{II}) \text{ cancelled and then squared.} \\
\dfrac{a^2 - x^2}{a^2 + c^2} = \dfrac{(b + d)^2}{2(b^2 + d^2)} & (\text{IV}) & = (\text{III}) \text{ taken as in Art. 106 (VIII).} \\
a^2 - x^2 = \dfrac{(b + d)^2}{2(b^2 + d^2)}(a^2 + c^2) & (\text{V}) & = (\text{IV}) \times (a^2 + c^2).
\end{array}
$$

$$x^2 = a^2 - \frac{(b + d)^2}{2(b^2 + d^2)}(a^2 + c^2) = \frac{2a^2(b^2 + d^2) - (b + d)^2 a^2 - (b + d)^2 c^2}{2(b^2 + d^2)}$$

$$= \frac{a^2(2b^2 + 2d^2 - b^2 + 2bd - d^2) - c^2(b + d)^2}{2(b^2 + d^2)} = \frac{a^2(b^2 - 2bd + d^2) - c^2(b + d)^2}{2(b^2 + d^2)}$$

$$= \frac{a^2(b - d)^2 - c^2(b + d)^2}{2(b^2 + d^2)}$$

NOTE.—In equations of the form of Ex. 5, in which the unknown quantity does not enter into both sides, the principles deduced in Art. 106 may be used with much advantage, as is here illustrated.

EXERCISE XLIX.

Find the values of x in the following equations :—

1. $2x^2 - 6 = x^2 + 3$.

2. $\dfrac{9}{2 + 2x} + \dfrac{9}{2 - 2x} = 25$.

3. $\dfrac{2x}{3} = \dfrac{x^2 + 3}{2x}$.

4. $4x^2 - 8x^0 = 1$.

5. $(x - 3)^2 = 13 - 6x$.

6. $3(x + 5)^2 - 7x = 23x$.

7. $\dfrac{10x^2 + 17}{18} - \dfrac{5x^2 - 4}{9} = \dfrac{12x^2 + 2}{11x^2 - 8}$.

8. $24 - \sqrt{9 + 2x^2} = 15$.

9. $a + \sqrt{(x - 3)(x + 3)} = 4a$.

10. $\dfrac{a}{x} + \dfrac{\sqrt{a^2 - x^2}}{x} = \dfrac{x}{b}$.

11. $\sqrt{a^2x^{-2} + b^2} - \sqrt{a^2x^{-2} - b^2} = b$.

12. $2ax^2 + b - 4 = cx^2 - 5 + d - ax^2$.

13. $\sqrt{a^2 - x^2} + x\sqrt{a^2 - 1} = a^2\sqrt{1 - x^2}$.

14. $x + \sqrt{b^2 + x^2} = \dfrac{cb^2}{\sqrt{b^2 + x^2}}$.

15. $\sqrt{3 + \frac{1}{4}x} - \sqrt{\frac{3}{4}x} = \sqrt{\frac{1}{4}x - 3}$. 16. $\sqrt[3]{a + x} + \sqrt[3]{a - x} = b$.

---

201. By transposition and reduction, and change of signs, if necessary, every affected quadratic equation may be reduced to the form

$$x^2 + px + q = 0$$

where p and q are either positive or negative, integral or fractional.

202. To investigate a rule for solving affected quadratic equations, we proceed as follows:—

If we take any binomial, as $x + a$, and square it, we obtain $x^2 + 2ax + a^2$. Now we observe that (a^2) the last term of this square is *the square of half the coefficient* of x in the second term, and we hence conclude that when we have reduced a given quadratic equation to the form $x^2 + px = -p$, we may regard the left-hand member as being composed of the first two terms of the square of a binomial, and that we may make the 1st member a complete square by adding to it the square of half the coefficient of its second term, and of course adding this to one side we must also add it to the other, in order to preserve the equality of the members. Thus we get

$$x^2 + px + \frac{p^2}{4} = -q + \frac{p^2}{4}.$$

The first member of this equation is now a complete square, and we observe that by extracting the square root of each side we shall get rid of the second power of the unknown quantity, and thus reduce the quadratic to a simple equation. Thus,

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q}.$$

Whence by transposition $x = -\frac{1}{2}p \pm \sqrt{\frac{p^2}{4} - q}$

That is, $x = \frac{1}{2}(\pm\sqrt{p^2 - 4q} - p)$

203. Hence for the solution of quadratic equations we have the following

RULE.—*By transposition and reduction arrange the equation in such a manner that the two terms involving the unknown quantities shall be alone on the left-hand side, and the coefficient of* x^2 *shall be* + 1.

II. *Add to each side of the equation the square of half the coefficient of* x.

III. *Extract the square root of both sides of the equation, and thence by transposition find the values of* x.

Ex. 1. Given $x^2 + 10x = -24$ to find the values of x.

OPERATION.

$x^2 + 10x = -24$	(I)	
$x^2 + 10x + 25 = 1$	(II)	= (I) with $(\frac{10}{2})^2 = 5^2 = 25$ added to each side.
$x + 5 = \pm 1$	(III)	= (II) with square root taken.
$x = \pm 1 - 5 = -4$ or -6	(IV)	= (III) transposed.

NOTE.—When we solved the general equation $x^2+px+q=0$, we obtained $x = \frac{1}{2}(\pm\sqrt{p^2-4q} - p)$. Now we may use this as a formula for finding the value of x in a quadratic equation. Thus, in the last example $p = 10$ and $q = 24$; then

$$x = \tfrac{1}{2}(\pm\sqrt{p^2-4q} - p) = \tfrac{1}{2}(\pm\sqrt{100-96} - 10) = \tfrac{1}{2}(\pm\sqrt{4} - 10)$$
$$= \tfrac{1}{2}(\pm 2 - 10) = \frac{-8}{2} \text{ or } \frac{-12}{2} = -4 \text{ or } -6.$$

But although quadratic equations may thus be solved by formula, this method should be resorted to only by the advanced student, as the junior student requires all the practice he can get in the solution of quadratics by completing the square, &c.

Ex. 2. Given $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{13}{6}$ to find the values of x.

OPERATION.

$\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{13}{6}$	(I)	
$6x^2 + 6(x+1)^2 = 13x(x+1)$	(II)	= (I) cleared of fractions.
$6x^2 + 6x^2 + 12x + 6 = 13x^2 + 13x$	(III)	= (II) expanded.
$x^2 + x = 6$	(IV)	= (III) transp. and red.
$x^2 + x + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$	(V)	= (IV) with $\frac{1}{4} = (\frac{1}{2})^2$ added.
$x + \frac{1}{2} = \pm\frac{5}{2}$	(VI)	= (V) with sq. root taken.
$x = \pm\frac{5}{2} - \frac{1}{2} = 2$ or -3	(VII)	= (VI) transposed and red.

Ex. 3. Given $\frac{2x+9}{9} + \frac{4x-3}{4x+3} = 3 + \frac{3x-16}{18}$ to find the values of x.

OPERATION.

$\frac{2x+9}{9} + \frac{4x-3}{4x+3} = 3 + \frac{3x-16}{18}$	(I)	
$4x + 18 + \frac{72x-54}{4x+3} = 54 + 3x - 16$	(II)	= (I) × 18.
$\frac{72x-54}{4x+3} = 20 - x.$	(III)	= (II) transp. and red.
$72x - 54 = 80x + 60 - 4x^2 - 3x$	(IV)	= (III) × $(4x + 3)$.
$4x^2 - 5x = 114$	(V)	= (IV) transp. and red.
$x^2 - \frac{5}{4}x = \frac{114}{4}$	(VI)	= (V) ÷ 4.
$x^2 - \frac{5}{4}x + \frac{25}{64} = \frac{114}{4} + \frac{25}{64} = \frac{1849}{64}$	(VII)	= (VI) with $(\frac{5}{8})^2$ added.
$x - \frac{5}{8} = \sqrt{\frac{1849}{64}} = \pm\frac{43}{8}$	(VIII)	= (VII) with square root of each side taken.
$x = \pm\frac{43}{8} + \frac{5}{8} = 6$ or $-4\frac{3}{4}$	(IX)	= (VIII) transp. and red.

Ex. 4. Given $\frac{a^2x^2}{m^2} - \frac{2ax}{c} = \frac{-m^2}{c^2}$ to find the value of x.

OPERATION.

$a^2c^2x^2 - 2acm^2x = -m^4$	(I)	
$x^2 - \frac{2m^2}{ac}x = -\frac{m^4}{a^2c^2}$	(II)	= I ÷ a^2c^2
$x^2 - \frac{2m^2}{ac}x + \frac{m^4}{a^2c^2} = 0$	(III)	= II with $\left(\frac{m^2}{ac}\right)^2$ added.
$x - \frac{m^2}{ac} = 0$	(IV)	= III with sq. root not taken.
$x = \frac{m^2}{ac}$	(V)	= IV transposed.

NOTE.—In this example we may conclude that the two roots of the equation are equal.

EXERCISE L.

Find the values of x in the following equations :—

1. $2x^2 + 8x - 20 = 70.$
2. $x^2 - 19 = 8x - 10.$
3. $x^2 - 8x = 20.$
4. $x^2 - 29 = 16 - 12x.$

5. $2x^2 + x - 15 = 70 - x - x^2$. 6. $x^2 - 4x + 15 = 10x - 2x^2$.

7. $118x - \frac{3}{2}x^2 = x^4 + 23\frac{1}{2}$. 8. $4x^2 - 3x - 20 = 5x + 300$.

9. $\frac{x}{a} + \frac{a}{x} = \frac{2}{a}$. 10. $x^2 + 3x - 72 = 201 - x - 4x^2$.

11. $\frac{3x}{x+2} - \frac{x-1}{6} = x - 2\frac{1}{2}$. 12. $\frac{x^2+12}{2} - 4x + \frac{1}{2}x = 0$.

13. $x^2 - x = \frac{8}{3}x - 2$. 14. $acx^2 + bcx = adx + bd$.

15. $\frac{x+\sqrt{x}}{x-\sqrt{x}} \quad \frac{x^2-x}{4}$. 16. $x^2 - x - 40 = 170$.

17. $\frac{x}{3} + \frac{3}{x} = \frac{x}{4} + \frac{4}{x} - \frac{11}{12}$. 18. $\frac{x-2}{x+2} - \frac{x-3}{x+3} = \frac{x+4}{x-4} - \frac{x+2}{x-2}$.

19. $(7x+3)(3+7x) = 10\{2(x-1)(3+x) - (3+2x)(x-3)\}$.

20. $ax^2 - bx + c = fx^2 + cx - b$.

21. $(a - m + x)^{-1} = a^{-1} - m^{-1} + x^{-1}$.

22. $abx^2 - 2x(a+b)\sqrt{ab} = (a-b)^2$.

204. Many of the foregoing equations when reduced assume the general form $ax^2 + bx + c = 0$, where a, b and c may be any quantities whatever; now when we further reduce this to bring it under the rule (Art. 203) we get $x^2 + \frac{b}{a}x = \frac{c}{a}$, and consequently we have the inconvenience of dealing with fractions throughout the entire process. To obviate this difficulty we may proceed as follows:—

Taking the equation $ax^2 + bx = c$, let us multiply every term by $4a$, and then add b^2 to each side of the resulting equation, and we get $4a^2x^2 + 4abx + b^2 = -4ac + b^2$. The left hand member is now a complete square, and extracting the square root of each member we get $2ax + b = \pm\sqrt{b^2 - 4ac}$

$$\text{whence } x = \frac{-b \pm \sqrt{b^2 - 4ac.}}{2a}$$

205. This operation translated gives us the following:—

RULE.—*Having reduced the equation to the form* $ax^2 + bx = c$, *multiply every term by four times the coefficient of* x^2, *and to each member of the resulting equation add the square of the coefficient of the second term.*

Then extract the square root of both terms, transpose and reduce and thus obtain the values of x.

Ex. 1. Given $3x^2 - 2x = 65$, to find the values of x.

OPERATION.

$3x^2 - 2x = 65$	(I)	
$36x^2 - 24x = 780$	(II)	= (I) × 12 i. e. 4 times 3, the coef. of x^2.
$36x^2 - 24x + 4 = 784$	(III)	= (II) with $(2)^2 = 4$ added to each side.
$6x - 2 = \pm 28$	(IV)	= (III) with square root extracted.
$6x = 2 \pm 28$	(V)	= (IV) transposed.
$6x = 30$ or -26	(VI)	= (V) reduced.
$x = 5$ or $-4\frac{1}{3}$	(VII)	= (V) ÷ 6.

Ex. 2. Given $\frac{3x - 7}{x} + \frac{4x - 10}{x + 5} = 3\frac{1}{2}$ to find the values of x.

OPERATION.

$\frac{3x-7}{x} + \frac{4x-10}{x+5} = 3\frac{1}{2}$	(I)	
$7x^2 - 39x = 70$	(II)	= (I) × $2x(x+5)$ and red.
$196x^2 - 1092x = 1960$	(III)	= (II) × 28 i. e. 4 times 7.
$196x^2 - 1092x + (39)^2 = 1960 + 1521$	(IV)	= (III) + $(39)^2$
$14x - 39 = \sqrt{3481} = \pm 59$	(V)	= $\sqrt{\text{IV}}$
$14x = 39 \pm 59 = 98$ or -20	(VI)	= (V) transposed.
$\therefore x = 7$ or $-1\frac{3}{7}$	(VII)	= (VI) ÷ 14.

Ex. 3. Given $(3a^2 + b^2)(x^2 - x + 1) = (3b^2 + a^2)(x^2 + x + 1)$ to find the values of x.

OPERATION.

$(3a^2 + b^2)(x^2 - x + 1) = (3b^2 + a^2)(x^2 + x + 1)$	(I)	
$\frac{x^2 - x + 1}{x^2 + x + 1} = \frac{3b^2 + a^2}{3a^2 + b^2}$	(II)	= (I) ÷ $(3a^2 + b^2)(x^2 + x + 1)$.
$\frac{2x^2 + 2}{-2x} = \frac{4b^2 + 4a^2}{2b^2 - 2a^2}$	(III)	= (II) as in Art. 106 (VII).
$\frac{x^2 + 1}{-x} = \frac{2b^2 + 2a^2}{b^2 - a^2}$	(IV)	= (III) reduced.
$(b^2 - a^2)x^2 + b^2 - a^2 = -2(b^2 + a^2)x$	(V)	= (IV) cleared of fractions.
$(b^2 - a^2)x^2 + 2(b^2 + a^2)x = a^2 - b^2$	(VI)	= (V) transposed.

$4(b^2 - a^2)^2x^2 + 8(b^4 - a^4)x + 4(b^2 + a^2)^2 = 4(a^2 - b^2)(b^2 - a^2) + 4(b^2 + a^2)^2$ (VII)

$4(b^2 - a^2)^2x^2 + 8(b^4 - a^4)x + 4(b^2 + a^2)^2 = 16a^2b^2$ (VIII)

$$2(b^2 - a^2)x + 2(b^2 + a^2) = \pm 4ab \text{ (IX)}$$
$$(b^2 - a^2)x + (b^2 + a^2) = \pm 2ab$$
$$(b^2 - a^2)x = -(b^2 + a^2) \pm 2ab = -a^2 \pm 2ab - b^2$$
$$(a^2 - b^2)x = a^2 \mp 2ab + b^2$$
$$x = \frac{(a-b)^2}{a^2 - b^2} \text{ or } \frac{(a+b)^2}{a^2 - b^2}$$
$$\therefore x = \frac{a-b}{a+b} \text{ or } \frac{a+b}{a-b}$$

(VII) = (VI) × 4 times coef. of x^2, i. e. × $4(b^2 - a^2)$ and then each side increased by the sq. of $2(b^2 + a^2)$, the coef. of the 2nd term. (VIII) = (VII) with right-hand member reduced. (IX) = $\sqrt{\text{VIII}}$.

EXERCISE LI.

Find the value of x in the following equations :—

1. $3x^2 - 9 = 76 - 2x$.
2. $x^2 - x = 210$.
3. $4x^2 - 3x = 85$.
4. $\frac{x}{5} + \frac{5}{x} = 5\frac{1}{5}$.
5. $4x^2 + 6x = 2x - x^2 + 273$.
6. $3x^2 + 8x + 11 = 32 - x^2$.
7. $5\frac{1}{2} - \frac{2}{x} = \frac{7}{x+1}$.
8. $a^2x^2 + abx = acx + bc$.
9. $\frac{1}{2}x^2 + 5 = \frac{3}{4}x + 5\frac{5}{8}$.
10. $7x^2 - 2 = -(2 - \sqrt{3})x + 4x^2\sqrt{3}$.
11. $x^2 + 6ax = b^2$.
12. $\frac{5-x}{3+x} - \frac{x}{3} = \frac{1}{9}x - \frac{7+4x}{19}$.
13. $\frac{x}{m} + \frac{m}{x} = \frac{5}{m}$.
14. $mx^2 + mn = 2mx\sqrt{n} + nx^2$.
15. $(1+x+x^2)^2 = \frac{(a+1)(1+x^2+x^4)}{a-1}$
16. $\frac{x^4+3x^3+6}{x^2+x-4} = x^2 + 2x + 15$.

THEORY OF QUADRATIC EQUATIONS.

206. We have seen (Art. 204) that the roots of the general equation $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{a}$$

Now from this it appears that

I. The two roots are real and different in value if $b^2 > 4ac$.

II. The two roots are real and equal in value if $b^2 = 4ac$.

III. The two roots are impossible or imaginary if $b^2 < 4ac$.

Hence if any equation be expressed in the form of $ax^2 + bx + c = 0$, *its roots are* REAL *and* DIFFERENT, REAL *and* EQUAL, *or* IMAGINARY, *according as* $b^2 >, =$ or $< 4ac$; *and similarly if the equation be of the form* $x^2 + px + q = 0$, *its roots are* REAL *and* DIFFERENT, REAL *and* EQUAL, *or* IMAGINARY, *according as* $p^2 >, =,$ *or* $< 4q$.

207. THEOREM I.—*A quadratic equation cannot have more than two roots.*

DEMONSTRATION. For if it be possible let the quadratic equation $ax^2 + bx + c$ have *three* roots as β, γ and δ. Then

$a\beta^2 + b\beta + c = 0$	(I)	
$a\gamma^2 + b\gamma + c = 0$	(II)	
$a\delta^2 + b\delta + c = 0$	(III)	
$a(\beta^2 - \gamma^2) + b(\beta - \gamma) = 0$	(IV)	= (I) − (II).
$a(\beta^2 - \delta^2) + b(\beta - \delta) = 0$	(V)	= (I) − (III).
$a(\beta - \gamma) + b = 0$	(VI)	= (IV) ÷ $(\beta - \gamma)$ which is not = 0, ∵ by hypothesis β is not = γ.
$a(\beta - \delta) + b = 0$	(VII)	= (V) ÷ $(\beta - \delta)$ which is not = 0, ∵ by hyp. β is not = δ.
$a(\gamma - \delta) = 0$	(VIII)	= (VII) − (VI).

Now a is not $= 0$, otherwise $ax^2 + bx + c = 0$ would become $bx + c = 0$, which is not a quadratic equation; therefore $(\gamma - \delta)$ must $= 0$, and therefore $\gamma = \delta$; but by hypothesis γ is not $= \delta$, which is absurd. Hence a quadratic equation cannot have three roots.

208. THEOREM II.—*In any quadratic equation reduced to the form of* $x^2 + px + q = 0$, *the coefficient of the 2nd term is equal, when its sign is changed, to the sum of the roots, and the 3rd term is equal to the product of the roots.*

DEMONSTRATION. Let the two roots of the equation $x^2 + px + q = 0$ be β and γ. Then $-\frac{1}{2}p + \sqrt{(\frac{1}{4}p^2 - q)} = \beta$

And $-\frac{1}{2}p - \sqrt{(\frac{1}{4}p^2 - q)} = \gamma$

By addition $-p = \beta + \gamma =$ sum of the roots.

By multiplication $\{-\frac{1}{2}p + \sqrt{(\frac{1}{4}p^2 - q)}\}\{-\frac{1}{2}p - \sqrt{(\frac{1}{4}p^2 - q)}\} = \beta\gamma$.

That is, $\frac{1}{4}p^2 - (\frac{1}{4}p^2 - q)$ which is $= q = \beta\gamma =$ product of roots.

Cor. If β and γ are the roots of the equation $ax^2 + bx + c = 0$, then
$$\beta + \gamma = -\frac{b}{a} \text{ and } \beta\gamma = \frac{c}{a}.$$

209. THEOREM III.—*If* β *and* γ *are the roots of the equation* $x^2 + px + q = 0$, *then* $(x - \beta)(x - \gamma) = x^2 + px + q$.

DEMONSTRATION. $(x - \beta)(x - \gamma) = x^2 - (\beta + \gamma)x + \beta\gamma$.

But $(\beta + \gamma) = -p$ and $\beta\gamma = q$. (By Art. 208.)

$\therefore (x - \beta)(x - \gamma) = x^2 - (-p)x + q = x^2 + px + q$.

Cor. If β, γ are the roots of the equation $ax^2 + bx + c = 0$, that is, of the equation $a(x^2 + \frac{b}{a}x + \frac{c}{a}) = 0$. Then we have
$$ax^2 + bx + c = 0 = a(x - \beta)(x - \gamma).$$

Cor. 2. If $ax^3 + bx^2 + cx + d = 0$ be a cubic equation, and if its roots be β, γ, δ; then $(x - \beta)(x - \gamma)(x - \delta) = ax^3 + bx^2 + cx + d$.

ILLUSTRATIVE EXAMPLES.

Ex. 1. Form the equation whose roots are -3 and 4.

OPERATION.

Since $x = -3$, $x + 3 = 0$, and since $x = 4$, $x - 4 = 0$.

Then $(x + 3)(x - 4) = 0$, that is $x^2 - x - 12 = 0$.

Ex. 2. Form the equation whose roots are 2, -2, 3 and 0.

OPERATION.

$x - 2 = 0$, $x + 2 = 0$, $x - 3 = 0$, $x = 0$. Then we have

$(x - 2)(x + 2)(x - 3)x = (x^2 - 4)(x^2 - 3x) = x^4 - 3x^3 - 4x^2 + 12x = 0$.

Ex. 3. Form the equation whose roots are 1, -1, 3, -2, and $2 \pm \sqrt{7}$.

OPERATION.

$x - 1 = 0$, $x + 1 = 0$, $x - 3 = 0$, $x + 2 = 0$, $x - 2 - \sqrt{7} = 0$, and $x - 2 + \sqrt{7} = 0$.

Then $(x - 1)(x + 1)(x - 3)(x + 2)(x - 2 - \sqrt{7})(x - 2 + \sqrt{7}) = 0$,

that is, $(x^2 - 1)(x^2 - x - 6)(x^2 - 4x + 4 - 7) = 0$,

that is, $x^6 - 5x^5 - 6x^4 + 32x^3 + 23x^2 - 27x - 18 = 0$.

Ex. 4. Find, without solving the equation, the sum, difference, and product of the roots of $x^2 - 42x + 117 = 0$.

OPERATION.

Let β and γ be the roots, then Art. 208 $\beta + \gamma = 42$ and $\beta\gamma = 117$. Then by inspection find two numbers whose sum $= 42$ and product $= 117$, and they are evidently 3 and 39, and hence the difference of the roots $= 36$.

Ex. 5. For what value of c^2m will the equation $3x^2 + 7x + c^2m = 0$ have equal roots?

OPERATION.

From Art. 206 it appears that in the equation $ax^2 + bx + c = 0$ the roots will be real and equal when $b^2 = 4ac$, that is, in this equation when $7^2 = 4 \times 3 \times c^2m$, or when $12c^2m = 49$, or $c^2m = 4\frac{1}{12}$.

Ex. 6. If β and γ be the roots of the equation $x^2 - px + q = 0$, find the value in terms of p and q of $\frac{\beta}{\gamma} + \frac{\gamma}{\beta}$, and of $\beta^3 + \gamma^3$.

OPERATION.

Art. 208. $\beta + \gamma = p$ and $\beta\gamma = q$.

Then $\frac{\beta}{\gamma} + \frac{\gamma}{\beta} = \frac{\beta^2 + \gamma^2}{\beta\gamma} = \frac{\beta^2 + \gamma^2}{\beta\gamma} + 2 - 2 = \frac{\beta^2 + 2\beta\gamma + \gamma^2}{\beta\gamma} - 2$ $= \frac{(\beta + \gamma)^2}{\beta\gamma} - 2 = \frac{p^2}{q} - 2 = \frac{p^2 - 2q}{2}$.

And $\beta^3 + \gamma^3 = \beta^3 + 3\beta^2\gamma + 3\beta\gamma^2 + \gamma^3 - (3\beta^2\gamma + 3\beta\gamma^2) = (\beta + \gamma)^3 - 3\beta\gamma(\beta + \gamma) = p^3 - 3qp = p(p^2 - 3q)$.

Exercise LII.

1. Form the equation whose roots are -2, and -7.
2. Form the equation whose roots are $4, -2, 1$, and 0.
3. Form the equation whose roots are $2, -2, 3, -3$, and 0.
4. Form the equation whose roots are $5, -5, 2, -2$, and $3 \pm \sqrt{2}$.
5. Form the equation whose roots are $1, 2, 3, 4$, and $5 \pm \sqrt{6}$.
6. Form the equation whose roots are $5, 4, 1, 0$, and $2 \pm \sqrt{-3}$.
7. Given 5 and -2, two roots of the equation $x^4 - 6x^3 + 5x^2 + 12x = 60$, to find the other roots.
8. Given $1 \pm \sqrt{-6}$, two roots of the equation $x^4 - 4x^3 + 8x^2 - 8x = 21$, to find the other roots.

9. Given 14, one root of the equation $x^3 + 6x^2 - 3920 = 0$, to find the other roots.

10. Given 2, one root of the equation $x^4 - 6x^3 + 13x^2 - 10x = 0$ to find the other roots.

11. Given 3 and -4, two roots of the equation $x^5 - 2x^4 - 25x^3 + 26x^2 + 120x = 0$, to find the other roots.

12. Given $\pm\sqrt{-2}$, two roots of the equation $x^5 - x^4 + 2x^2 - 4x = 0$, to find the other roots.

13. For what value of c will the equation $2x^2 + 4x + c = 0$ have equal roots.

14. If β and γ be the roots of the equation $ax^2 + bx + c = 0$, form the equation whose roots are the reciprocals of these.

15. If β and γ be the roots of the equation $x^2 + px + q = 0$, find the value of $\beta^2 + \gamma^2$, of $(\beta - \gamma)^2$; of $\beta^2 - \gamma^2$; of $\frac{1}{\beta} + \frac{1}{\gamma}$ and of $\beta^3 - \gamma^3$.

EQUATIONS WHICH MAY BE SOLVED LIKE QUADRATICS.

210. There are many equations which though not quadratics in reality may be solved by the rules for quadratics. Such, among others, are equations which come under one or other of the general forms $ax^{2n} + bx^n + c = 0$ or $ax^{\frac{2}{n}} + bx^{\frac{1}{n}} + c = 0$, in which n is any integral number, and a, b, c, positive or negative, integral or fractional.

Ex. 1. Given $x + 6x^{\frac{1}{2}} = -8$ to find the values of x.

OPERATION.

$x + 6x^{\frac{1}{2}} = -8$	(I)	
$x + 6x^{\frac{1}{2}} + 9 = 1$	(II)	= (I) with square completed by adding 9 to each side.
$x^{\frac{1}{2}} + 3 = \pm 1$	(III)	= (II) with square root extracted.
$x^{\frac{1}{2}} = \pm 1 - 3$	(IV)	= (III) transposed.
$x^{\frac{1}{2}} = -2$ or -4	(V)	= (IV) reduced.
$x = 4$ or 16	(VI)	= (V) squared.

Ex. 2. Given $\sqrt[3]{x^2} + 22\sqrt[3]{x} = 23$ to find the values of x.

OPERATION.

$x^{\frac{2}{3}} + 22x^{\frac{1}{3}} = 23$	(I)	
$x^{\frac{2}{3}} + 22x^{\frac{1}{3}} + 121 = 144$	(II)	= (I) with $(11)^2$ added to each side.
$x^{\frac{1}{3}} + 11 = \pm 12$	(III)	= (II) with square root extracted.
$x^{\frac{1}{3}} = 1$ or -23	(IV)	= (III) transposed and reduced.
$x = 1$ or -12167	(V)	= (IV) cubed.

Ex. 3. Given $\sqrt{x+12} + \sqrt[4]{x+12} = 6$ to find the values of x.

OPERATION.

$(x+12)^{\frac{2}{4}} + (x+12)^{\frac{1}{4}} = 6$	(I)	
$(x+12)^{\frac{2}{4}} + (x+12)^{\frac{1}{4}} + \frac{1}{4} = \frac{25}{4}$	(II)	= (I) with $\frac{1}{4}$ added to each side
$(x+12)^{\frac{1}{4}} + \frac{1}{2} = \pm \frac{5}{2}$	(III)	= (II) with sq. root taken.
$(x+12)^{\frac{1}{4}} = 2$ or -3	(IV)	= (III) transposed and reduced.
$x + 12 = 16$ or 81	(V)	= (IV) raised to 4th power.
$x = 4$ or 69	(VI)	= (V) transposed and reduced.

Ex. 4. Given $x^6 - 35x^3 = -216$ to find the values of x.

OPERATION.

$x^6 - 35x^3 = -216$	(I)	
$4x^6 - 140x^3 + 1225 = 361$	(II)	= (I) $\times$ 4 and $(35)^2$ added.
$2x^3 - 35 = \pm 19$	(III)	= (II) with sq. root taken.
$2x^3 = 54$ or 16	(IV)	= (III) transposed and reduced.
$x^3 = 27$ or 8	(V)	= (IV) $\div$ 2.
$x = 3$ or 2	(VI)	= (V) with $\sqrt[3]{}$ taken.

Ex. 5. Given $5\sqrt{(x^2 + 5x + 28)} = x^2 + 5x + 4$ to find the values of x.

OPERATION.

$x^2 + 5x + 4 - 5\sqrt{(x^2 + 5x + 28)} = 0$	(I)	
$(x^2 + 5x + 28) - 5(x^2 + 5x + 28)^{\frac{1}{2}} = 24$	(II)	= (I) with 24 added to each side.
$(x^2+5x+28) - 5(x^2+5x+28)^{\frac{1}{2}} + \frac{25}{4} = \frac{121}{4}$	(III)	= (II) with $(\frac{5}{2})^2$ added.
$(x^2 + 5x + 28)^{\frac{1}{2}} - \frac{5}{2} = \pm \frac{11}{2}$	(IV)	= (III) with $\sqrt{}$ taken.

$(x^2+5x+28)^{\frac{1}{2}} = 8$ or -3	(V)	= (IV) transp. and red.
$x^2+5x+28 = 64$ or 9	(VI)	= (V) squared.
$x^2+5x = 36$ or -19	(VII)	= (VI) transp. and red.
$x^2+5x+\frac{25}{4} = \frac{169}{4}$ or $-\frac{51}{4}$	(VIII)	= (VII) with $(\frac{5}{2})^2$ added to
$x+\frac{5}{2} = \pm\frac{13}{2}$ or $\pm\frac{1}{2}\sqrt{-51}$	(IX)	= (VIII) with sq. root taken
$x = 4$ or -9; or $\frac{1}{2}(-5 \pm\sqrt{-51})$	(X)	= (IX) transp. and red.

NOTE.—In this example we should find by trial that only the first two roots, i. e. 4 and -9 are roots of the proposed equation, the other two being roots of the equation $x^2+5x+4+5\sqrt{(x^2+5x+28)} = 0$.

Ex. 6. Given $\dfrac{(5x^4+10x^2+1)(5a^4+10a^2+1)}{(x^4+10x^2+5)(a^4+10a^2+5)} = ax$ to find the values of x.

OPERATION.

$\dfrac{(5x^4+10x^2+1)(5a^4+10a^2+1)}{(x^4+10x^2+5)(a^4+10a^2+5)} = ax$	(I)	
$\dfrac{5x^4+10x^2+1}{x^5+10x^3+5x} = \dfrac{a^5+10a^3+5a}{5a^4+10a^2+1}$	(II)	$= (\text{I}) \times \dfrac{1}{x} \times \dfrac{a^4+10a^2+5}{5a^4+10a^2+1}$
$\dfrac{x^5+5x^4+10x^3+10x^2+5x+1}{x^5-5x^4+10x^3-10x^2+5x-1} = \dfrac{1+5a+10a^2+10a^3+5a^4+a^5}{1-5a+10a^2-10a^3+5a^4-a^5}$	(III)	= (II) taken thus: $\dfrac{Den.+Num.}{Den.-Num.} = \dfrac{Den.+Num.}{Den.-Num.}$
$\dfrac{(x+1)^5}{(x-1)^5} = \dfrac{(1+a)^5}{(1-a)^5}$	(IV)	= (III) bracketed.
$\dfrac{x+1}{x-1} = \dfrac{1+a}{1-a}$	(V)	= (IV) with $\sqrt[5]{}$ taken.
$\dfrac{2x}{2} = \dfrac{2}{2a}$	(VI)	= (V) taken as in (III) above
$x = \dfrac{1}{a}$	(VII)	= (VI) cancelled.

Ex. 7. Given $x^6-1=0$ to find the values of x.

OPERATION.

$x^6-1=0$	(I)	
$(x^3+1)(x^3-1)=0$	(II)	= (I) factored.
$x^3+1=0$	(III)	Equation (II) is satisfied by taking either $x^3-1=0$ or $x^3+1=0$, and therefore we consider $x^3-1=$ one root and $x^3+1=$ other root, and we get separately $x^3+1=0$ and $x^3-1=0$.
$x^3-1=0$	(IV)	

$$
\begin{array}{r|c|l}
(x+1)(x^2-x+1)=0 & (\text{v}) & =(\text{iii})\text{ factored.} \\
(x-1)(x^2+x+1)=0 & (\text{vi}) & =(\text{iv})\text{ factored.} \\
x+1=0 & (\text{vii}) & =\text{one factor of }(\text{v}). \\
x^2-x+1=0 & (\text{viii}) & =\text{other factor of }(\text{v}). \\
x-1=0 & (\text{ix}) & =\text{one factor of }(\text{vi}). \\
x^2+x+1=0 & (\text{x}) & =\text{other factor of }(\text{vi}).
\end{array}
$$

$$\therefore\ x=1,\ x=-1,\ x=\tfrac{1}{2}(1\pm\sqrt{-3})\text{ and }x=\tfrac{1}{2}(-1\pm\sqrt{-3}).$$

NOTE.—Nos. (VII) and (IX) give us by transposition $x=-1$ and $x=1$, and solving the quadratic equations (VIII) and (X) we get the other four roots $x=\frac{1}{2}(1\pm\sqrt{-3})$ and $x=\frac{1}{2}(-1\pm\sqrt{-3})$.

The above is of course equivalent to finding the six, sixth roots of unity.

Ex. 8. Given $x^4+x^3-4x^2+x+1=0$ to find the values of x.

OPERATION.

$$
\begin{array}{l|c|l}
x^4+x^3-4x^2+x+1=0 & (\text{i}) & \\
x^2+x-4+\dfrac{1}{x}+\dfrac{1}{x^2}=0 & (\text{ii}) & =(\text{i})\div x^2. \\
x^2+\dfrac{1}{x^2}+x+\dfrac{1}{x}=4 & (\text{iii}) & =(\text{ii})\text{ transposed and arranged.} \\
\left(x^2+2+\dfrac{1}{x^2}\right)+\left(x+\dfrac{1}{x}\right)=6 & (\text{iv}) & =(\text{iii})\text{ with 2 added to each side.} \\
\left(x+\dfrac{1}{x}\right)^2+\left(x+\dfrac{1}{x}\right)=6 & (\text{v}) & =(\text{iv})\text{ differently expressed.} \\
\left(x+\dfrac{1}{x}\right)^2+\left(x+\dfrac{1}{x}\right)+\tfrac{1}{4}=\tfrac{25}{4} & (\text{vi}) & =(\text{v})\text{ with sq. completed by adding }\tfrac{1}{4}\text{ to each side.} \\
\left(x+\dfrac{1}{x}\right)+\tfrac{1}{2}=\pm\tfrac{5}{2} & (\text{vii}) & =(\text{vi})\text{ with }\sqrt{\ }\text{ taken.} \\
x+\dfrac{1}{x}=2\text{ or }-3 & (\text{viii}) & =(\text{vii})\text{ transposed and reduced.}
\end{array}
$$

Thus we get two distinct quadratic equations :—

I. $x+\dfrac{1}{x}=2$ or $x^2-2x=-1$ whence $x=1$;

II. $x+\dfrac{1}{x}=-3$ or $x^2+3x=-1$ whence $x=\frac{1}{2}(-3\pm\sqrt{5})$.

Ex. 9. Given $x^3+3x=14$ to find the values of x.

OPERATION.

$x^3 + 3x = 14$	(I)	
$x^4 + 3x^2 = 14x$	(II)	$=$ (I) $\times x$.
$x^4 + 7x^2 = 4x^2 + 14x$	(III)	$=$ (II), $4x^2$ added to each side.
$x^4 + 7x^2 + \frac{49}{4} = 4x^2 + 14x + \frac{49}{4}$	(IV)	$=$ (III) with sq. completed by adding $\frac{49}{4}$ to each side.
$x^2 + \frac{7}{2} = \pm(2x + \frac{7}{2})$	(V)	$=$ (IV) with $\sqrt{}$ taken.

This gives us two separate quadratic equations :—

I. $x^2 + \frac{7}{2} = 2x + \frac{7}{2}$ or $x^2 - 2x = 0$ whence $x = 2$ or 0; and

II. $x^2 + \frac{7}{2} = -2x - \frac{7}{2}$ or $x^2 + 2x = -7$ whence $x = -1 \pm \sqrt{-6}$.

Ex. 10. Given $\dfrac{49x^2}{4} + \dfrac{48}{x^2} - 49 = 9 + \dfrac{6}{x}$ to find the values of x.

OPERATION.

$\dfrac{49x^2}{4} + \dfrac{48}{x^2} - 49 = 9 + \dfrac{6}{x}$	(I)	
$\dfrac{49x^2}{4} - 49 + \dfrac{48}{x^2} = \dfrac{6}{x} + 9$	(II)	$=$ (I) arranged.
$\dfrac{49x^2}{4} - 49 + \dfrac{49}{x^2} = \dfrac{1}{x^2} + \dfrac{6}{x} + 9$	(III)	$=$ (II) with $\dfrac{1}{x^2}$ added.
$\dfrac{7x}{2} - \dfrac{7}{x} = \pm\left(\dfrac{1}{x} + 3\right)$	(IV)	$=$ (III) with $\sqrt{}$ taken.

This also gives us two distinct quadratic equations :—

I. $\dfrac{7x}{2} - \dfrac{7}{x} = \dfrac{1}{x} + 3$ or $7x^2 - 6x = 16$ whence $x = 2$ or $-1\frac{1}{7}$; and

II. $\dfrac{7x}{2} - \dfrac{7}{x} = -\dfrac{1}{x} - 3$ or $7x^2 + 6x = 12$ whence $x = \frac{1}{7}(-3 \pm \sqrt{93})$.

EXERCISE LIII

Find the values of x in the following equations :—

1. $x - 6\sqrt{x} = 16$.
2. $x^{\frac{1}{2}} - 4x^{\frac{1}{4}} = -3$.
3. $x^4 + 20 = 14x^2 - 20$.
4. $x^3 + 7\sqrt{x^3} = 1107 - 7x^{\frac{3}{2}}$.
5. $x - 3\sqrt{x + 6} = 2 - \sqrt{x + 6}$.
6. $2x^4 - x^2 = 496$.
7. $x^6 - 8x^3 = 513$.
8. $x + 5 = 6 + \sqrt{x + 5}$.

9. $\sqrt{x^5} + \sqrt{x^3} = 6\sqrt{x}$. 10. $\dfrac{\sqrt{4x}+2}{4+\sqrt{x}} = \dfrac{4-\sqrt{x}}{\sqrt{x}}$.

11. $\sqrt[4]{x+21} = 12 - \sqrt{x+21}$. 12. $\sqrt{x^3} - 2\sqrt{x} - x = 0$.

13. $\dfrac{x^5 + x^4 + 2}{x^5 - x^4} = \dfrac{x^3 + x^2 - 2}{x^3 - x^2}$.

14. $\dfrac{54 - 9\sqrt{x}}{x + 2\sqrt{x}} = \dfrac{7x^2 - 3x + 4}{(6+\sqrt{x})(x + 2\sqrt{x})} + \dfrac{23x - 46\sqrt{x}}{6 + \sqrt{x}}$.

15. $x^4 - 3x^2 + 3x = 9$.

16. $\sqrt{(x-1)(x-2)} + \sqrt{(x-3)(x-4)} = \sqrt{2}$.

17. $x^3 - 3x + 2 = 0$.

18. $\sqrt{x^2 + ax + b} + \sqrt{x^2 - ax + b} = c$.

19. $\dfrac{x}{\sqrt{x} + \sqrt{a-x}} + \dfrac{x}{\sqrt{x} - \sqrt{a-x}} = \dfrac{b}{\sqrt{x}}$.

20. $\sqrt{x+60} + \sqrt{x^2+9} = \dfrac{2\sqrt{x^3 + 60x^2 + 9x + 540} + 89}{\sqrt{x+60} + \sqrt{x^2+9}}$

21. $x^{12} = 1$.

22. $x^3 - 6x^2 + 11x = 6$.

23. $x^3 - 4x^2 + x + 6 = 6$.

24. $x^3 - 8x^2 + 11x = -20$.

25. $\dfrac{x+a}{x+b} = \left(\dfrac{2x+a+c}{2x+b+c}\right)^2$.

26. $3x^3 - 14x^2 + 21x = 10$.

27. $x + a + 3\sqrt[3]{abx} = b$.

28. $9x - 4x^2 + (4x^2 - 9x + 11)^{\frac{1}{2}} = 5$.

29. $(x+6)^2 + 2x^{\frac{1}{2}}(x+6) = 138 + \sqrt{x}$.

30. $x^4 - 4x^3 + 6x^2 - 4x = 5$.

31. $2x\sqrt{1-x^4} = a(1+x^4)$.

32. $\{(x-2)^2 - x\}^2 - (x-2)^2 = 88 - (x-2)$.

33. $ax^4 + bx^3 + cx^2 + bx + a = 0$.

34. $\sqrt{\left(x^2 - \dfrac{a^4}{x^2}\right)} + \sqrt{\left(a^2 - \dfrac{a^4}{x^2}\right)} = \dfrac{x^2}{a}$.

35. $\sqrt{(2x+4)} - 2\sqrt{(2-x)} = \dfrac{12x - 8}{\sqrt{(9x^2 + 16)}}$.

36. $\dfrac{2x^2 + 1 + x\sqrt{(4x^2+3)}}{2x^2 + 3 + x\sqrt{(4x^2}} =$

37. $(x-1)(x-2)(x-3)(x-4)=8$.

38. $(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)$
$=(x^2-9x)(17x^2-153x+230)+401$.

39. $(x-1)(x-2)(x-3)=(x+1)(x+2)(x+3)$.

40. $(\sqrt{x+1}-2)(\sqrt{x+1}-3)+5\sqrt{\{\sqrt{x+1}(\sqrt{x+1}-6)+\sqrt{x+1}-1\}}=0$.

41. $8x^4-16x^3+4x^2-x-2(2x^2-2x+1)\sqrt{4x^4-8x^3-4x^2+3x-1}=0$.

42. $abx^{-2}+\dfrac{2(a+x)(a^2c^{-1}x^2-b)}{ax}=c^{-1}(x^3-\dfrac{bcx}{a^2}+a^3)$.

43. $8x^3+22x^2+24x+9=0$.

44. $3x^4-4x^3+17x^2-6x=-5$.

45. $\dfrac{x^2+2x(\sqrt{3}-\sqrt{5})-\frac{2}{3}\sqrt{135}+8}{x-\sqrt{3}+\sqrt{5}}-\dfrac{x^2-2x(\sqrt{3}-\sqrt{5})-\sqrt{2}(\sqrt{30}-\sqrt{32})}{x+\sqrt{3}-\sqrt{5}}$
$=x^2-8-\sqrt{15}$.

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE.

211. No general rule can be given for the solution of quadratic equations involving more than one unknown quantity. In dealing with these therefore the student must be left very much to his own ingenuity. Very often by attentively considering the question an artifice will suggest itself, by means of which the roots may be easily found. The following solutions afford illustrations of the employment of artifices which are very frequently used with much advantage.

Ex. 1. Given $\left.\begin{array}{l} x^2-y^2=51 \\ x+y=17 \end{array}\right\}$ to find the values of x and y.

OPERATION.

$x^2-y^2=51$	(I)	
$x+y=17$	(II)	
$x-y=3$	(III)	$=$ (I) $\div$ (II).
$2x=20$	(IV)	$=$ (II) $+$ (III).
$x=10$	(V)	$=$ (IV) $\div 2$.
$2y=14$	(VI)	$=$ (II) $-$ (III).
$y=7$	(VII)	$=$ (V) $\div 2$.

Ex. 2. Given $\left.\begin{aligned} x^2 + y^2 &= 74 \\ x + y &= 12 \end{aligned}\right\}$ to find the values of x and y.

OPERATION.

$x^2 + y^2 = 74$	(I)	
$x + y = 12$	(II)	
$x^2 + 2xy + y^2 = 144$	(III)	= (II) squared.
$2xy = 70$	(IV)	= (III) − (I).
$x^2 - 2xy + y^2 = 4$	(V)	= (I) − (IV).
$x - y = 2$	(VI)	= (V) with $\sqrt{}$ taken.
$2x = 14 \therefore x = 7$	(VII)	= (II) + (VI).
$2y = 10 \therefore y = 5$	(VIII)	= (II) − (VI).

Or thus:—

$x^2 + y^2 = 74$	(I)	
$x + y = 12$	(II)	
$x = 12 - y$	(III)	= (II) transposed.
$x^2 = (12 - y)^2$	(IV)	= (III) squared.
$(12 - y)^2 + y^2 = 74$	(V)	= (I) with $(12-y)^2$ subs. for x^2.
$144 - 24y + y^2 + y^2 = 74$	(VI)	= (V) expanded.
$2y^2 - 24y = -70$	(VII)	= (VI) transposed.
$y^2 - 12y = -35$	(VIII)	= (VII) ÷ 2.
$y^2 - 12y + 36 = 1$	(IX)	= (VIII) with sq. completed by adding 36 to each side.
$y - 6 = \pm 1$	(X)	= (IX) with $\sqrt{}$ taken.
$y = 7$ or 5	(XI)	= (X) transposed.

Then $x = 12 - y = 12 - 7$ or $12 - 5 = 5$ or 7.

Ex. 3. Given $\left.\begin{aligned} x + y &= 33 \\ xy &= 266 \end{aligned}\right\}$ to find the values of x and y.

OPERATION.

$x + y = 33$	(I)	
$xy = 266$	(II)	
$x^2 + 2xy + y^2 = 1089$	(III)	= (I) squared.
$4xy = 1064$	(IV)	= (II) × 4.
$x^2 - 2xy + y^2 = 25$	(V)	= (III) − (IV).
$x - y = \pm 5$	(VI)	= (V) with $\sqrt{}$ taken.
$2x = 38$ or $28 \therefore x = 19$ or 14	(VII)	= (I) + (VI).
$2y = 28$ or $38 \therefore y = 14$ or 19		= (I) − (VI).

Or thus:

$x + y = 33$	(I)	
$xy = 266$	(II)	
$x = 33 - y$	(III)	= (I) transposed.
$y(33 - y) = 266$	(IV)	= (II) with $33 - y$ sub. for x.
$y^2 - 33y = -266$	(V)	= (IV) expanded and $\times -1$.
$4y^2 - 132y + (33)^2 = 25$	(VI)	= (V) $\times$ 4 and with 1089 added to each side.
$2y - 33 = \pm 5$	(VII)	= (VI) with $\surd$ taken.
$2y = 38$ or 28	(VIII)	= (VII) transposed.
$y = 19$ or 14	(IX)	= (VIII) $\div$ 2.

Ex. 4. Given $\left.\begin{array}{r} 2x^2 + 3xy + y^2 = 20 \\ 5x^2 + 4y^2 = 41 \end{array}\right\}$ to find the values of x.

OPERATION.

In equations like this, in which either or both of the equations are *homogeneous* in all those terms which involve these quantities, put $x = vy$, then $x^2 = v^2y^2$, and $xy = vy^2$, and the solution will be much facilitated.

$2x^2 + 3xy + y^2 = 20$	(I)	
$5x^2 + 4y^2 = 41$	(II)	
$2v^2y^2 + 3vy^2 + y^2 = 20$	(III)	= (I) with vy written for x.
$5v^2y^2 + 4y^2 = 41$	(IV)	= (II) with vy subs. for x.
$(2v^2 + 3v + 1)y^2 = 20$	(V)	= (III) factored.
$(5v^2 + 4)y^2 = 41$	(VI)	= (IV) factored.
$y^2 = \dfrac{20}{2v^2 + 3v + 1}$	(VII)	= (V) $\div (2v^2 + 3v + 1)$.
$y^2 = \dfrac{41}{5v^2 + 4}$	(VIII)	= (VI) $\div (5v^2 + 4)$.
$\dfrac{20}{2v^2 + 3v + 1} = \dfrac{41}{5v^2 + 4}$	(IX)	= right hand members of (VII) and (VIII) equated to one another (Ax. XI).
$6v^2 - 41v = -13$	(X)	= (IX) reduced.
$v = \frac{1}{3}$ or $\frac{13}{2}$	(XI)	= (X) solved by ordinary rule

$y^2 = \dfrac{41}{5v^2 + 4} = \dfrac{41}{5(\frac{1}{3})^2 + 4}$ or $\dfrac{41}{5(\frac{13}{2})^2 + 4} = 9$ or $\frac{4}{21}$. Hence $y = 3$ or $\sqrt{\frac{4}{21}} = \frac{2}{21}\sqrt{21}$.

$$x = vy = \tfrac{1}{3} \times 3 \text{ or } \tfrac{13}{2} \times \tfrac{2}{21}\sqrt{21} = 1 \text{ or } \tfrac{13}{21}\sqrt{21}.$$

Ex. 5. Given $\left.\begin{aligned} x^3 + y^3 &= 189 \\ x^2y + xy^2 &= 180 \end{aligned}\right\}$ to find the values of x and y.

OPERATION.

In order to show that several different plans may generally be adopted in dealing with simultaneous quadratics, so as to evolve the values of x and y, we shall give two or three different solutions of this problem.

1ST METHOD.

$$\begin{array}{rl|l}
x^3 + y^3 = 189 & \text{(I)} & \\
x^2y + xy^2 = 180 & \text{(II)} & \\
3x^2y + 3xy^2 = 540 & \text{(III)} & = \text{(II)} \times 3. \\
\hline
x^3 + 3x^2y + 3xy^2 + y^3 = 729 & \text{(IV)} & = \text{(I)} + \text{(III)} \\
x + y = 9 & \text{(V)} & = \text{(IV) with } \sqrt[3]{\ } \text{ taken.} \\
xy(x + y) = 180 & \text{(VI)} & = \text{(II) factored.} \\
xy = 20 & \text{(VII)} & = \text{(VI)} \div \text{(V)}
\end{array}$$

Hence $x = 9 - y$; $xy = y(9 - y) = 20$ or $y^2 - 9y = -20$, whence $y = 5$ or 4 and $x = 4$ or 5.

2ND METHOD.

$$\begin{array}{rl|l}
x^3 + y^3 = 189 & \text{(I)} & \\
x^2y + xy^2 = 180 & \text{(II)} & \\
xy\,(x + y) = 180 & \text{(III)} & = \text{(II) factored.} \\
x + y = \dfrac{180}{xy} & \text{(IV)} & = \text{(III)} \div xy. \\
x^3 + 3x^2y + 3xy^2 + y^3 = \dfrac{180^3}{x^3y^3} & \text{(V)} & = \text{(IV) raised to 3rd power} \\
3x^2y + 3xy^2 = \dfrac{180^3}{x^3y^3} - 189 & \text{(VI)} & = \text{(V)} - \text{(I)}. \\
3xy(x + y) = \dfrac{5832000 - 189x^3y^3}{x^3y^3} & \text{(VII)} & = \text{(VI) simplified} \\
xy(x + y) = \dfrac{1944000 - 63x^3y^3}{x^3y^3} & \text{(VIII)} & = \text{(VII)} \div 3. \\
180 = \dfrac{1944000 - 63x^3y^3}{x^3y^3} & \text{(IX)} & = \text{(VIII) with 180 substituted} \\
 & & \quad \text{for } xy\,(x + y). \\
180x^3y^3 = 1944000 - 63x^3y^3 & \text{(X)} & = \text{(IX) cleared of fractions.} \\
243x^3y^3 = 1944000 & \text{(XI)} & = \text{(X) transposed.} \\
x^3y^3 = 8000 & \text{(XII)} & = \text{(XI)} \div 243. \\
xy = 20 & \text{(XIII)} & = \text{(XII) with } \sqrt[3]{\ } \text{ taken.}
\end{array}$$

Then, as before, since $xy(x+y) = 180$ and $xy = 20$ $\therefore x + y = 9$ and $x = 9 - y$, whence $y(9 - y) = 20$ or $y^2 - 9y = -20$, wherefore $y = 5$ or 4 and $x = 4$ or 5.

3RD METHOD.

$x^3 + y^3 = 189$	(I)	
$x^2y + xy^2 = 180$	(II)	
$(v+z)^3 + (v-z)^3 = 189$	(III)	= (II) with $(v+z)$ written for x and $(v-z)$ for y.
$2v(v^2 - z^2) = 180$	(IV)	= (III) written thus, $xy(x+y)$ and then $(v+z)$ and $v-z$ substituted for x and y
$2v^3 + 6vz^2 = 189$	(V)	= (III) expanded and red.
$2v^3 - 2vz^2 = 180$	(VI)	= (IV) expanded.
$6v^3 - 6vz^2 = 540$	(VII)	= (VI) × 3.
$8v^3 = 729$ or $2v = 9$ or $v = \frac{9}{2}$	(VIII)	= (V) + (VII)
$8vz^2 = 9$ or $8z^2 \times \frac{9}{2} = 9$ or $z = \pm\frac{1}{2}$	(IX)	= (V) − (VI).

Hence $x = v + z = \frac{9}{2} \pm \frac{1}{2} = 5$ or 4.

$y = v - z = \frac{9}{2} - (\pm\frac{1}{2}) = \frac{9}{2} \mp \frac{1}{2} = 4$ or 5.

4TH METHOD.

$x^3 + y^3 = 189$	(I)	
$x^2y + xy^2 = 180$	(II)	
$xy(x+y) = 180$	(III)	= (II) factored
$x + y = \dfrac{180}{xy}$	(IV)	= (III) ÷ xy.
$x^2 - xy + y^2 = \dfrac{189xy}{180}$	(V)	= (I) ÷ (IV).
$v^2y^2 - vy^2 + y^2 = \dfrac{189vy^2}{180}$	(VI)	= (V) with vy subs. for x.
$180v^2y^2 - 180vy^2 + 180y^2 = 189vy^2$	(VII)	= (IV) × 180.
$20v^2 - 41v + 20 = 0$	(VIII)	= (VII) trans. and ÷ $9y^2$.
$20v^2 - 41v = -20$	which	is a quadratic equation, whence $v = \frac{5}{4}$ or $\frac{4}{5}$.
$v^2y^3 + vy^3 = 180$ or $y^3 = \dfrac{180}{v^2 + v}$	(IX)	= (II) with vy subs. for x.

Hence $y^3 = \dfrac{180}{\frac{25}{16} + \frac{5}{4}}$ or $\dfrac{180}{\frac{16}{25} + \frac{4}{5}} = 64$ or 125 whence $y = 4$ or 5 and $x = 5$ or 4.

In order to save figures, the second method is better applied by letting $x + y = s$ and $xy = p$, then

$x^3 + y^3 = 189$	(I)	
$x^2y + xy^2 = 180$	(II)	
$s^3 - 3sp = 189$	(III)	$\because x^3 + y^3 = (x+y)^3 - 3xy(x+y)$.
$sp = 180$	(IV)	$\because x^2y + xy^2 = xy(x+y)$.
$s = \dfrac{180}{p}$	(V)	$=$ (III) $\div p$.
$s^3 = \dfrac{180^3}{p^3}$	(VI)	$=$ (V) cubed.
$3sp = \dfrac{180^3}{p^3} - 189$	(VII)	$=$ (V) $-$ (III).
$540 = \dfrac{180^3}{p^3} - 189$	(VIII)	$=$ (IV) $\times 3$ and subs. for left-hand member.
$729 = \dfrac{180^3}{p^3}$	(IX)	$=$ (VIII) transposed.
$9 = \dfrac{180}{p}$	(X)	$=$ (IX) with $\sqrt[3]{\ }$ taken.
$p = 20$	(XI)	$=$ (X) $\times p$ and $\div 9$.
$sp = 180 \therefore s = 9$	(XII)	$=$ (IV) with value of p. subs.

Hence $p = xy = 20$, and $s = x + y = 9$, &c.

EXERCISE LIV.

Find the values of x and y in the following equations :—

1. $\left.\begin{array}{r} x^2 - y^2 = 45 \\ x - y = 5 \end{array}\right\}$ 2. $\left.\begin{array}{r} x^2 - y^2 = 105 \\ x + y = 21 \end{array}\right\}$ 3. $\left.\begin{array}{r} x^2 + y^2 = 41 \\ x + y = 9 \end{array}\right\}$

4. $\left.\begin{array}{r} x^2 + y^2 = 113 \\ x - y = 15 \end{array}\right\}$ 5. $\left.\begin{array}{r} x^2 + y^2 = 89 \\ xy = 40 \end{array}\right\}$ 6. $\left.\begin{array}{r} x^2 - y^2 = 55 \\ 3xy = 72 \end{array}\right\}$

7. $\left.\begin{array}{r} x^2 + 3y^2 = 148 \\ 2x + y = 24 \end{array}\right\}$ 8. $\left.\begin{array}{r} 3x^2 - 2y^2 = 115 \\ 2x - 3y = 2 \end{array}\right\}$ 9. $\left.\begin{array}{r} 4x^2 + 3y^2 = 511 \\ 3x + 2y = 27 \end{array}\right\}$

10. $\left.\begin{array}{r} x^3 - y^3 = 26 \\ x - y = 2 \end{array}\right\}$ 11. $\left.\begin{array}{r} x + y = 4 \\ x^3 + y^3 = (x+y)^2 \end{array}\right\}$ 12. $\left.\begin{array}{r} \sqrt[3]{x} + \sqrt[3]{y} = 3 \\ \sqrt[3]{xy} = 2 \end{array}\right\}$

13. $\left.\begin{array}{r} x + 4y = 14 \\ y^2 + 4x = 2y + 11 \end{array}\right\}$ 14. $\left.\begin{array}{r} 2x^2 + xy - 5y^2 = 20 \\ 2x - 3y = 1 \end{array}\right\}$

15. $\left.\begin{array}{r} \dfrac{9x + 5y}{4} = xy \\ x - y = 2 \end{array}\right\}$ 16. $\left.\begin{array}{r} x^2y^2 + 4xy = 96 \\ x + y = 6 \end{array}\right\}$

17. $\left.\begin{aligned} \frac{x^2}{y^2} + \frac{4x}{y} &= \frac{85}{9} \\ x - y &= 2 \end{aligned}\right\}$

18. $\left.\begin{aligned} x^2 + xy &= 77 \\ xy - y^2 &= 12 \end{aligned}\right\}$

19. $\left.\begin{aligned} x^2 + xy &= 66 \\ x^2 - y^2 &= 11 \end{aligned}\right\}$

20. $\left.\begin{aligned} \frac{x^2}{y} + \frac{y^2}{x} &= 18 \\ x + y &= 12 \end{aligned}\right\}$

21. $\left.\begin{aligned} x^5 + y^5 &= 3368 \\ x + y &= 8 \end{aligned}\right\}$

22. $\left.\begin{aligned} x^3 + y^3 &= 133 \\ x + y &= 7 \end{aligned}\right\}$

23. $\left.\begin{aligned} x^4 + y^4 &= 97 \\ x - y &= 1 \end{aligned}\right\}$

24. $\left.\begin{aligned} x^3 + y^3 &= 91 \\ x^2y + xy^2 &= 84 \end{aligned}\right\}$

25. $\left.\begin{aligned} \frac{x + y}{xy} &= \frac{3}{4} \\ x + y - 13 &= 13 - x^2 - y^2 \end{aligned}\right\}$

26. $\left.\begin{aligned} \frac{x + y}{x - y} + \frac{x - y}{x + y} &= \frac{26}{5} \\ x^2 + y^2 &= 52 \end{aligned}\right\}$

27. $\left.\begin{aligned} x + y &= x^2 \\ 7y - 2x &= 36 \end{aligned}\right\}$

28. $\left.\begin{aligned} x^4 + y^4 &= 14x^2y^2 \\ x + y &= m \end{aligned}\right\}$

29. $\left.\begin{aligned} x^2 + 2y^2 &= 74 - xy \\ 2xy + y^2 &= 73 - 2x^2 \end{aligned}\right\}$

30. $\left.\begin{aligned} x^4 - x^2 + y^4 - y^2 &= 84 \\ x^2 + 2x^2y^2 + y^2 &= 85 \end{aligned}\right\}$

31. $\left.\begin{aligned} 3x^2 + 2xy - 4y^2 &= 108 \\ x^2 - 3xy - 7y^2 &= -81 \end{aligned}\right\}$

32. $\left.\begin{aligned} y^2 - x^2 - y - x &= 12 \\ (y - x)^2(y + x) &= 48 \end{aligned}\right\}$

33. $\left.\begin{aligned} \frac{x^2}{y^2} + \frac{2x + y}{\sqrt{y}} &= 20 - \frac{y^2 + x}{y} \\ x + 8 &= 4y \end{aligned}\right\}$

34. $\left.\begin{aligned} x^3 + y^3 &= 35 \\ x^2 + y^2 &= 13 \end{aligned}\right\}$

35. $\left.\begin{aligned} \frac{\sqrt{(y^2+1)}+1}{\sqrt{(y^2+1)}-1} &= \frac{\sqrt{(x+9)}+3}{\sqrt{(x+9)}-3} \\ x\,(y + 1)^2 &= 36(y^3 + 1\tfrac{7}{9}) \end{aligned}\right\}$

36. $\left.\begin{aligned} x^4 + y^4 &= x \\ x^3 + y^3 &= 1 \end{aligned}\right\}$

37. $\left.\begin{aligned} (x^6 + 1)y &= (y^2 + 1)x^3 \\ (y^6 + 1)x &= 9(x^2 + 1)y^3 \end{aligned}\right\}$

38. $\left.\begin{aligned} \frac{x^2}{y^2} + \frac{y}{x} + \frac{x}{y} &= \frac{27}{4} - \frac{y^2}{x^2} \\ x - y &= 2 \end{aligned}\right\}$

39. $\left.\begin{aligned} \sqrt{(5\sqrt{x} + 5\sqrt{y})} + \sqrt{y} &= 10 - \sqrt{x} \\ \sqrt{x^5} + \sqrt{y^5} &= 275 \end{aligned}\right\}$

40. $\left.\begin{aligned} x^3 + y^3 &= x - y \\ x^2 + y^2 &= axy \end{aligned}\right\}$

41. $\left.\begin{aligned} xy + a(x - y) &= a^2 \\ x + y^2 + a^3 &= 0 \end{aligned}\right\}$

42. $\left.\begin{array}{l} x^2 + y^2 + a^2 = 0 \\ x^4 + y^4 + a^4 + x^2(3y^2 + a^2) = 0 \end{array}\right\}$

43. $\left.\begin{array}{l} x^2 + 3y + a^3 = 0 \\ x^6 - 3y^3 + a^6 + x^2y(3x^2 - y) = a^3x^2(x^2 + 2) \end{array}\right\}$

44. $\left.\begin{array}{l} x - y = a \\ x^4 + y^4 = b^4 \end{array}\right\}$

45. $\left.\begin{array}{l} x^2 - xy + y^2 = a^2 \\ x^4 - x^2y^2 + y^4 = b^4 \end{array}\right\}$

46. $\left.\begin{array}{l} 3x^6 - 12x^4 + 18x^2 = 2y^6 - 11y^4 + 52y^2 + 27 \\ x^4 - y^4 - 3 + 2x^2(a-1) = 2a(y^2-1) + 2y^2(x^2-1) \end{array}\right\}$ to find x and y independent of a

47. $\left.\begin{array}{l} (y^2-x^2)(y^2-x^2+4)+5 = 2\sqrt{4(y^6-x^6)-(5x^2+12x^2y^2-5y^2)(y^2-x^2)} \\ y^4 - 3y^2 - 1 = 5x^2 - 8x(1 - \sqrt{x^2 - 2x + 5}) + 4 \end{array}\right\}$

48. $\left.\begin{array}{l} (x^2 - y^2)(x^2 + y^2 - 4) = 4(x^2 - 3) \\ x^2y^2 + 7(x^2 - y^2) = 6xy\sqrt{y^2 - x^2} \end{array}\right\}$

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

1. What two numbers are those whose difference is 5 and the product of whose sum by the greater is 228?

SOLUTION.

Let x = the greater, then $x - 5$ = the less.

$x + x - 5 = 2x - 5$ = their sum.

Then $x(2x - 5) = 228$	(I)	
$2x^2 - 5x = 228$	(II)	
$16x^2 - 40x + 25 = 1849$	(III)	= (II) × 8, then sq. completed.
$4x - 5 = \pm 43$	(IV)	= (III) with $\sqrt{}$ taken.
$4x = 48$ or -38		

$\therefore x = 12$ or $-9\frac{1}{2}$ = the greater.

$x - 5 = 7$ or $-14\frac{1}{2}$ = the less.

2. A poulterer bought 15 ducks and 12 turkeys for 105 shillings, at the rate of 2 ducks more for 18 shillings than of turkeys for 20 shillings. What was the price of each?

SOLUTION.

Let x = price of a duck in shillings and y = price of a turkey.

Then $15x + 12y = 105$	(I)	
$\frac{18}{x} = \frac{20}{y} + 2$	(II)	
$5x + 4y = 35$	(III)	= (I) reduced.
$9y - 10x = xy$	(IV)	= (II) reduced.
$10x + 8y = 70$	(V)	= (III) × 2.
$17y = xy + 70$	(VI)	= (IV) + (V).
$x = \frac{35 - 4y}{5}$	(VII)	= (III) transposed and reduced.
$17y - y\left(\frac{35 - 4y}{5}\right) = 70$	(VIII)	= (VI) with $\frac{35 - 4y}{5}$ subs. for x.
$2y^2 + 25y = 175$	(IX)	= (VIII) reduced.
$16y^2 + 200y + 625 = 2025$	(X)	= (IX) × 8 and sq. complete.

$4y + 25 = \pm 45.$

$4y = 20$ or -70 whence $y = 5$s.

$x = \frac{35 - 4y}{5} = \frac{35 - 20}{5} = 3$s.

NOTE.—The negative value -17s. 6d. for the price of a turkey is not taken into account here, as although $-17\frac{1}{2}$ is undoubtedly a root of the equation $2y^2 + 25y = 175$, yet -17s. 6d. as the price of a turkey does not satisfy the conditions of the problem as given and must therefore be neglected.

3. Find a number such that the sum of its square and its cube shall be nine times the next higher number.

SOLUTION.

Let x = the number, then x^2 = its square, and x^3 = its cube; also $x + 1$ = the next higher number.

Then $x^3 + x^2 = 9(x + 1)$	(I)	
$x^2(x + 1) = 9(x + 1)$	(II)	= (I) factored.
$x^2 = 9$	(III)	= (II) ÷ $x + 1$.
$x = \pm 3$	(IV)	= (III) with $\sqrt{\ }$ taken.

Verification. Take $+3$; then $27 + 9 = 36 = 9(3 + 1)$.

Take -3; then $-27 + 9 = -18 = 9(-3 + 1) = 9 \times -2$.

4. A person at play won, at the first game, as much money as he had in his pocket; at the second game he won 5 shillings more than the square root of what he then had; at the third game he won the square of all that he then had, and he found that he then possessed £112 16s. What had he at first?

SOLUTION.

Let x = the shillings he had at first.

Then $2x$ = the shillings he had at the end of the 1st game.

$\sqrt{2x} + 5$ = sum won at the 2nd game.

$2x + \sqrt{2x} + 5$ = sum at end of 2nd game.

$(2x + \sqrt{2x} + 5)^2$ = sum won at 3rd game.

$(2x + \sqrt{2x} + 5)^2 + 2x + \sqrt{2x} + 5)$ = sum at the end of the 3rd game. Then

$(2x+\sqrt{2x}+5)^2+(2x+\sqrt{2x}+5) = 2256$	(I)	
$(2x+\sqrt{2x}+5)^2+(2x+\sqrt{2x}+5)+\frac{1}{4} = \frac{9025}{4}$	(II)	= (I) with $\frac{1}{4}$ added.
$(2x+\sqrt{2x}+5)+\frac{1}{2} = \pm\frac{95}{2}$	(III)	= (II) with $\sqrt{\ }$ taken.
$2x+\sqrt{2x} = 42$ or -53	(IV)	= (III) transposed.
Rejecting the negative result we	have	
$(2x)+\sqrt{2x} = 42$	(V)	
$(2x)+\sqrt{2x}+\frac{1}{4} = \frac{169}{4}$	(VI)	= (V) with sq. comp.
$\sqrt{2x}+\frac{1}{2} = \pm\frac{13}{2}$	(VII)	= (VI) with $\sqrt{\ }$ taken
$\sqrt{2x} = 6$ or -7	(VIII)	= (VII) transposed.
$2x = 36$ or 49	(IX)	= (VIII) squared.
$x = 18$s.	(X)	= (IX) $\div$ 2.

NOTE.—The $24\frac{1}{2}$ which we get here as one value of x is not admissible as an answer to the problem, simply because it does not answer the conditions of the problem as given, and it obviously arises from the fact that the $\sqrt{2x}$ may be either $\pm$. It becomes an answer of the problem if we understand that at the 2nd game he *lost* a sum which was 5 shillings less than the square root of what he then had.

5. What number is that which being divided by the product of its digits, the quotient is 2, and if 27 be added to the number, the digits will be inverted?

SOLUTION.

Let x and y = the digits, x being the left-hand one.

Then $10x + y$ = the number, and xy = the product of the digits

$\left.\begin{array}{l}\dfrac{10x+y}{xy} = 2 \\ 10x+y+27 = 10y+x\end{array}\right\}$	(I) (II)	
$x = y - 3$	(III)	= (II) reduced and transposed.
$10x + y = 2xy$	(IV)	= (I) × xy.
$10(y-3)+y = 2y(y-3)$	(V)	= (IV) with $y - 3$ subs. for x.
$2y^2 - 17y = -30$	(VI)	= (V) reduced and transposed.
$16y^2 - 136y + (17)^2 = 49$	(VII)	= (VI) × 8 and with sq. complete.
$4y - 17 = \pm 7$	(VIII)	= (VII) with $\sqrt{}$ taken.

$4y = 24$; $y = 6$; $x = y - 3 = 6 - 3 = 3$

Hence the required number is 36.

NOTE.—The second value of y is obviously not admissible here.

6. A and B travelled on the same road and at the same rate to London. At the 50th milestone from London A overtook a flock of geese, which travelled at the rate of 3 miles in 2 hours, and 2 hours afterwards he met a waggon which travelled at the rate of 9 miles in 4 hours. B overtook the flock of geese at the 45th milestone from London, and met the waggon 40 minutes before he came to the 31st milestone. Where was B when A reached London?

SOLUTION.

A and B travel in the same direction, at the same rate, and on the same road, and consequently the distance between them is always the same.

Let x = rate per hour of travelling.

The places where A and B overtook the geese are 5 miles apart, and as the geese travel at the rate of $\frac{3}{2}$ of a mile per hour, to travel over 5 miles they would require $5 \div \frac{3}{2} = \frac{10}{3}$ hours. But in $\frac{10}{3}$ hours A has moved on $\frac{10x}{3}$ miles, while the geese have moved on only 5 miles.

Therefore distance in miles between A and $B = \frac{10x}{3} - 5$.

Again, A met the waggon $50 - 2x$ miles from London, while B met it $31 + \frac{2x}{3}$ miles from London, consequently as the waggon was travelling *from* London, the distance in miles travelled by the waggon between the two meeting was $\left(31 + \frac{2x}{3}\right) - (50 - 2x) = \frac{8x - 57}{3}$ miles. And since the waggon travelled at the rate of $\frac{9}{4}$ miles per hour, $\frac{8x - 57}{3} \div \frac{9}{4} = \frac{32x - 228}{27}$ = time in hours which elapsed between the meeting.

But in $\frac{32x - 228}{27}$ hours A has moved toward London $\left(\frac{32x - 228}{27}\right)x$ miles while the waggon has gone in the opposite direction $\left(\frac{8x - 57}{3}\right)$ miles.

Therefore distance in miles between A and $B = \frac{32x^2 - 228x}{27} + \frac{8x - 57}{3}$.

And since distance between A and B is always the same,

$\frac{32x^2 - 228x}{27} + \frac{8x - 57}{3} = \frac{10x}{3} - 5$	(I)	
$16x^2 - 123x = 189$	(II)	= (I) reduced.
$1024x^2 - 7872x + (123)^2 = 27225$	(III)	= (II) × 64 and with sq. then completed.
$32x - 123 = 165$	(IV)	= (III) with $\sqrt{}$ taken.

$$x = \frac{165 + 123}{32} = 9 = \text{rate per hour of travelling.}$$

Distance of B from $A = \frac{10x}{3} - 5 = \frac{90}{3} - 5 = 25$ miles = distance of B from London when A arrives there.

EXERCISE LV.

1. Divide the number 19 into two parts such that their product shall be 84.

2. What two numbers are those whose sum = 17, and the product of whose difference by the greater is 30.

3. There is a rectangular field whose area is 2080 rods, and its length exceeds its breadth by 12 rods. Required its dimensions.

4. What two numbers are those whose difference is 9, and the sum of whose squares is 353 ?

5. Divide the 16 into two parts such that their product added to the sum of their squares shall be 208.

6. A commission merchant sold a quantity of wheat for $171, and gained as much per cent. as the wheat cost him. What was the price of the wheat ?

7. A person bought a number of sheep for $80, and found that if he had bought 4 more for the same sum they would have each cost $1 less. How many did he buy ?

8. A certain number consisting of three digits is such that the sum of the squares of the digits, without considering their position, is 104, and the square of the middle digit exceeds twice the product of the other two by 4 ; also if 594 be subtracted from the number its digits will be inverted. Required the number.

9. A farmer paid $240 for a certain number of sheep, out of which he reserved 15, and sold the remainder for $216, gaining 40 cents a-head on those he sold. How many sheep did he buy, and what was the price of each ?

10. What two numbers are those whose sum is 10, and the sum of whose cubes is 280 ?

11. What are the two parts of 24 whose product is equal to 35 times their difference.

12. Find two numbers such that their sum, their product, and the difference of their squares are all equal to one another.

13. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards, but if the circumference of each had been increased one yard, the fore-wheel would have made only 4 revolutions more than the hind-wheel in going the same distance. What is the circumference of each wheel ?

14. The sum of two fractions is $1\frac{14}{15}$ and the sum of their reciprocals is $2\frac{5}{12}$. What are the two fractions ?

15 A person dies leaving $46800 to be divided equally among his children. It chances, however, that immediately after the

death of the father two of his children also die, and in consequence of this each remaining child receives $1950 more than it was entitled to by the father's will. How many children were there?

16. During the time that the shadow of a sun-dial which shows true time, moves from one o'clock to five, a clock which is too fast by a certain number of hours and minutes, strikes a number of strokes, which is equal to that number of hours and minutes, and it is observed that the number of minutes is less by 41 than the square of the number which the clock strikes at the last time of striking. The clock does not strike 12 during the time. How much is it too fast?

17. Two locomotives commence running at the same time from the two extremities of a railroad 324 miles in length; one travelling 3 miles an hour faster than the other, and they meet after having travelled as many hours as the slower travelled miles per hour. Required the distance travelled by each.

18. A person ordered $144 to be distributed among some poor people; but, before the money was divided there came in two claimants more by which means the share of each was $1 below what it would otherwise have been. What was the number at first?

19. Find a number such that, being divided by the product of its two digits the quotient is 2; and 27 being added to the number its digits are inverted.

20. A grocer sold 60 lbs. of coffee and 80 lbs. of sugar for $25, but he sold 24 lbs. more of sugar for $8 than he did of coffee for $10. What was the price of a lb. of each?

21. A and B engage to cradle a field of grain for $36, and as A alone could cradle it in 18 days, they promise to complete it in 10 days. They found however that they were obliged to call in C, an inferior workman, to assist them for the last four days, in consequence of which B received $1·50 less than he would otherwise have done. In what time could B or C separately reap the field?

22. A rectangular vat 5 feet deep holds, when filled to the depth of 4 feet, less than when completely filled by a number of cubic feet equal to 80, together with half the number of feet in

the perimeter of the base. It is also observed that the length of a pole, which reaches from one of the corners of the top to the opposite corner of the bottom of the vat, is equal to $\frac{3}{10}$ of the number of feet in the square inscribed on the diagonal of the bottom. Required the dimensions of the vat.

23. Two persons set out at the same time to travel on foot, A from Toronto to Cobourg, and B from Cobourg to Toronto. When they meet it is found that A has travelled 15 miles more than B, and that A will reach Cobourg in 2 hours; and B, Toronto in $4\frac{1}{2}$ hours after they have met. Find the distance between Toronto and Cobourg and the rate of travelling of each.

24. Find two numbers such that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

25. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for $\frac{2}{3}$ of the time that Silenus would have taken to empty the whole cask. Silenus then awoke and drank what Bacchus had left. Had they drank both together it would have been emptied two hours sooner, and Bacchus would have drank only half what he left Silenus. How long would it have taken each to empty the cask separately?

SECTION X.

RATIO, PROPORTION, AND VARIATION.

RATIO.

212. Ratio is the relation one quantity bears to another in regard to magnitude, the comparison being made by considering what multiple or fraction the first is of the second.

NOTE.—It will be seen from this definition that the term *ratio* is equivalent to the common arithmetical term *quotient*.

213. The ratio of one quantity to another is expressed by placing a colon between them or by writing them in the form of a fraction.

Thus, the ratio of a to b is written $a : b$ or more commonly $\frac{a}{b}$

214. Ratio can exist, of course, only between quantities of the same kind, because it is only between such quantities that any comparison as to magnitude can be instituted.

215. Quantities are of the same kind when one can be multiplied so as to exceed the other.

Thus, a ratio can exist between a cent and £100, or between a square nch and an acre, or between a grain troy and a cwt., because in each case the one can be multiplied so as to exceed the other, or, in other words the quantities entering into the ratio are of the same kind ; but no ratio can exist between a linear inch and an acre, because the former cannot be multiplied so as to exceed the latter.

216. The term of the ratio which precedes the sign : or which is written as numerator of the fraction is called the *antecedent* of the ratio, the remaining term, the *consequent*.

217. A ratio is said to be a *ratio of greater inequality*, *a ratio of equality*, or *a ratio of less inequality*, according as the antecedent is $>$, $=$, or $<$ the consequent.

218. If the antecedents of any ratios be multipled together and also the consequents, there is formed a new ratio which is said to be compounded of the former ratios.

Thus, the ratio $ace : bdf$ is said to be compounded of the ratios $a : b$, $c : d$, and $e : f$.

219. A ratio compounded of two ratios is called the sum of these ratios, thus, when the ratio $a : b$ is compounded with itself the resulting ratio $a^2 : b^2$ is called the double of the ratio $a : b$ or more commonly the *duplicat* ratio of $a : b$; also the ratio $a^3 : b^3$ is called the triple o

the ratio $a : b$ or more commonly the *triplicate* ratio of $a : b$.

NOTE.—Similarly the ratio $\sqrt{a} : \sqrt{b}$ is called *the subduplicate*, the ratio $\sqrt[3]{a} : \sqrt[3]{b}$, the subtriplicate; $a^{\frac{3}{2}} : b^{\frac{3}{2}}$, the sesquiplicate of the ratio $a : b$, &c.

220. Problems upon ratios are solved by writing the ratios as fractions and treating these fractions by the ordinary rules. Ratios are compared with one another as to magnitude by writing them as fractions, reducing these fractions to a common denominator and comparing the numerators.

221. THEOREM I.—*A ratio of greater inequality is diminished, and a ratio of less inequality increased by adding the same quantity to both its terms.*

DEMONSTRATION.—Let $a : b$ be a ratio of inequality, and let x be added to each term.

Then* $\frac{a}{b} \gtrless \frac{a+x}{b+x}$ as $ab + ax \gtrless ab + bx$, or as $ax \gtrless bx$ or as $a \gtrless b$. That is if $a > b$ then $ax > bx$ and $ab + ax > ab + bx$ and $\frac{a}{b} > \frac{a+x}{b+x}$; but if $a < b$ then $ax < bx$ and $ab + ax < ab + bx$ and $\frac{a}{b} < \frac{a+x}{b+x}$

* Read $\frac{a}{b}$ is *greater than or less than* $\frac{a+x}{b+x}$ according as, &c.

222. THEOREM II.—*A ratio of greater inequality is increased, and a ratio of less inequality diminished by subtracting the same quantity from both its terms.**

DEMONSTRATION.—Let $a : b$ be a ratio of inequality, and let x be subtracted from each term.

Then $\frac{a}{b} \gtrless \frac{a-x}{b-x}$, as $ab - ax \gtrless ab - bx$; or as $bx \gtrless ax$ or as $b \gtrless a$.

* The quantity subtracted must however be less than either of the terms.

223. *A ratio is increased or diminished by being compounded with another ratio according as the latter is a ratio of greater or less inequality.*

DEMONSTRATION.—Let the ratio $a : b$ be compounded with the ratio $m : n$, the latter being a ratio ot inequality.

Then $\frac{a}{b} \lessgtr \frac{am}{bn}$, according as $abn \lessgtr abm$, or as $n \lessgtr m$, or as $m : n$ is a ratio of greater or less inequality.

EXERCISE LVI.

1. Find the ratio compounded of $a : b$; $c : a^2$; and $ab : cd$.

2. Compound together the ratios $a^2 - b^2 : a^3 + b^3$; $(a - b)^2 : a$ and $a^2 - ab + b^2 : (a - b)^3$.

3. Compound together the ratios $x^2 - 2x - 15 : x^2 - 3x - 10$; $x^2 + x - 2 : x^2 + 8x + 15$ and $x^2 + 12x + 35 : x^2 - 1$.

4. Which is the greater ratio that of $a^3 + b^3 : a^2 + b^2$ or $a^2 + b^2 : a + b$.

5. Which is the greater ratio that of $x^2 + y^2 : x^2 - y^2$ or $(x + y)^4 : x^4 - x^3 y + x^2 y^2 - xy^3 + y^4$; $x\sqrt[3]{5}$ being $> y\sqrt[3]{7}$.

6. What quantity must be subtracted from each term of the ratio $a : b$ in order to make it equal to the ratio $c : d$.

7. What quantity must be added to each term of the ratio $m : n$ in order to convert it into a ratio of equality.

8. If $a : b$ be a ratio of greater inequality, what is the ratio compounded of the ratio of $a + b : a - b$, the difference of the duplicate ratios of $a : a$ and $a : b$, and the triplicate ratio of $b : a + b$.

9. Prove that the ratio $a : b$ is the duplicate of the ratio of $a + c$ to $b + c$, if c be a mean proportional between a and b.

10. Prove that $a^2 - b^2 : a^2 + b^2$ is greater or less than the ratio of $a - b : a + b$ according as $a : b$ is a ratio of greater or less inequality.

PROPORTION.

224. Proportion consists in an equality between two ratios, the two equal ratios being connected by the sign :: or by the ordinary sign of equality.

For example, if a, b, c, and d be four proportional quantities, the proportion existing between them is expressed by writing them thus, $a : b : : c : d$.

NOTE 1.—The first and fourth of such proportional quantities are called the *extremes*; and the second and third, the *means*.

NOTE 2.—When three quantities a, b and c, are proportionals, so that $a : b : : b : c$; the second term, b is said to be a *mean proportional* between the other two, and the third term c is called a *third proportional* to the other two.

225. THEOREM I.—*If four quantities be proportionals, the product of the extremes is equal to the product of the means.*

DEMONSTRATION.—Let $a : b : : c : d$, then $ad = bc$.

For $\frac{a}{b} = \frac{c}{d}$ and multiplying each of these by bd we have $ad = bc$.

COR. Hence if three terms of a proportion are given, the fourth may be readily found. Thus, $a = \frac{bc}{d}$; $b = \frac{ad}{c}$; $c = \frac{ad}{b}$; $d = \frac{bc}{a}$

226. THEOREM II.—*If the product of any two quantities be equal to the product of any two others, the four are proportionals—the factors of either product being made the extremes, and the factors of the other product the means.*

DEMONSTRATION.—Let $ad = bc$, then dividing each of these by bd and we have $\frac{a}{b} = \frac{c}{d}$ that is $a : b : : c : d$.

227. Since the two ratios composing a proportion may be written as two equal fractions, it follows that all the results obtained in Art. 106 may be applied to proportional quantities, or in other words, we may combine together in any manner whatever by addition or subtraction the first and second terms of a proportion, provided we similarly combine the third and fourth terms. So also we may proceed with any multiples whatever of the first and third, and any multiples whatever of the second and fourth terms. Similarly we may combine any powers or roots of the first and second terms, provided we also combine the same powers or roots of the third and fourth. (See the demonstrations in Art. 106 (I–XVI).

228. In solving problems in proportion the student must carefully bear the last proposition (227) in mind, and also that:—

I. Any proportion may be converted into an equation by taking the product of the extremes equal to the product of the means.

II. Any proportion may be converted into an equation, by writing the first term divided by the second = the third term divided by the fourth.

Ex. 1. If $a : b :: c : d$ prove that $(a+b)(c+d) = \frac{b}{d}(c+d)^2 = \frac{b}{d}(a+b)^2$.

Here $\frac{a}{b} = \frac{c}{d}$ $\therefore a = \frac{bc}{d}$ and $c = \frac{ad}{b}$

In the expression $(a+b)(c+d)$ substitute $\frac{bc}{d}$ for a, and we have $(a+b)(c+d) = \left(\frac{bc}{d} + b\right)(c+d) = \left(\frac{bc+bd}{d}\right)(c+d) = \frac{b}{d}(c+d)(c+d) = \frac{b}{d}(c+d)^2$

Similarly in the expression $(a+b)(c+d)$ substitute $\frac{ad}{b}$ for c.

This gives us $(a+b)(c+d) = (a+b)\left(\frac{ad}{b} + d\right) = (a+b)\left(\frac{ad+bd}{b}\right) = (a+b)(a+b)\frac{d}{b} = \frac{d}{b}(a+b)^2$.

Ex. 2.—Given $x^3+y^3 : x^3-y^3 : : 559 : 127$ and $x^2y = 294$ to find the values of x and y.

OPERATION.

$127x^3 + 127y^3 = 559x^3 - 559y^3$ or $686y^3 = 432x^3$ or $343y^3 = 216x^3$ or $7y = 6x$ $\therefore y = \frac{6}{7}x$. Substitute this value of y in the second equation and we have

$x^2y = 294$ or $x^2 \times \frac{6}{7}x = 294$ or $\frac{6x^3}{7} = 294$, or $\frac{x^3}{7} = 49$; or $x^3 = 343$; or $x = 7$, whence $y = 6$.

Ex. 3.—If $a : b :: c : d$ and also $m : n :: p : q$.

Prove that $ma + nb : ma - nb :: pc + qd : pc - qd$.

Since $a : b :: c : d$ and $m : n :: p : q$, then $\frac{a}{b} = \frac{c}{d}$ and $\frac{m}{n} = \frac{p}{q}$. Multiplying these equals together, we have $\frac{a}{b} \times \frac{m}{n} = \frac{c}{d} \times \frac{p}{q}$ or $\frac{ma}{nb} = \frac{pc}{qd}$. Then, Art. 106 (vii), $\frac{ma + nb}{ma - nb} = \frac{pc + qd}{pc - qd}$ that is $ma + nb : ma - nb :: pc + qd : pc - qd$.

Exercise LVII.

1. If a, b, c, d be any four quantities whatever, find what quantity added to each will make them proportionals.

2. If four numbers be proportionals show that there is no number which, being added to each will leave the resulting four numbers proportionals.

3. If $a : b :: c : d$ and $m : n :: p : q$ prove that $ma^2 - 2nb^2 : pc^2 - 2qd^2 :: ma^2 + 2nb^2 : pc^2 + 2qd^2$.

4. There are two numbers whose product is 24, and the difference of their cubes is to the cube of their difference as 19 to 1. What are the numbers?

5. The number 20 is divided into two parts, which are to each other in the duplicate ratio of 3 to 1. What is the mean proportional between these parts?

6. If $x : y :: a^3 : b^3$ and $a : b :: \sqrt[3]{c + x} : \sqrt[3]{d + y}$ prove that $dx = cy$.

7. If $(a + b + c + d)(a - b - c + d) = (a - b + c - d)(a + b - c - d)$ prove that $a : b :: c : d$.

8. What two numbers are those whose sum, difference and product are as the numbers s, d and p respectively.

9. A person in a railway carriage observes that another train running on a parallel line in the opposite direction occupies two seconds in passing him; but, if the two trains had been proceeding in the same direction, it would have required 30 seconds to pass him; compare the rates of the two trains.

10. A and B speculate in trade with different sums of money. A gains \$150 and B loses \$50, and now A's stock is to B's as

3 : 2, but had A lost \$50 and B gained \$100, A's stock would have been to B's as 5 : 9. What was the stock of each?

11. If $b = \sqrt{ac}$ prove that $a + b + c : (a + b + c)^2 : : a - b + c : a^2 + b^2 + c^2$.

12. If $b = \sqrt{ac}$ prove that $a : c : : (a+b)(a-b) : (b+c)(b-c)$.

13. What number is that to which if 3, 8 and 17 be severally added, the first sum shall be to the second as the second sum is to the third.

14. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings be as high as a pile of q sovereigns, compare the values of equal bulks of gold and silver.

15. If $a : b : : c : d$ prove that $\dfrac{\cdot 4\dot{2}a + 11\frac{1}{7}b}{4a - 5b} = \dfrac{\cdot 4\dot{2}c + 11\frac{1}{7}d}{4c - 5d}$

16. If a, b, c, and d are in continued proportion, express $(a + b)(c - d)$ in terms of a and c, and prove that $a : \sqrt[3]{a} : : b : \sqrt[3]{d}$.

VARIATION.

229. Variation is an abridged method of indicating proportion, and is conveniently used in investigating the relation which varying but dependent quantities bear to one another.

The two terms of a variation are the two antecedents of the corresponding proportion—the consequents not being expressed. Thus, when we say the interest varies as the principal, we mean that if P and p be any two principals and I and i, the corresponding interests at a given rate and time, then

$I : i : : P : p$ or briefly, omitting the consequents, $I \propto P$.

230. The sign $\propto$ is called the *sign of variation* and is read *varies as*.

Thus, $I \propto P$ is read, *I varies as P*.

231. One quantity is said to *vary directly* as another when the two quantities depend upon each other, so that if one be changed in any manner the other must also be changed in the same proportion.

Thus, leaving time and rate per cent. out of consideration, the interest (I) varies directly as the principal (P), for if I is changed to i, P must also be changed to p in such a manner that $I : i :: P : p$.

NOTE.—When we simply say that one quantity *varies* as another, we are always understood to mean that the one varies *directly* as the other.

232. One quantity is said to *vary inversely* as another when the first cannot be changed in any manner, but the *reciprocal* of the second is changed in the same proportion.

Thus, $A \propto \frac{I}{B}$ (A varies inversely as B), if, when A is changed to a, B must be changed to b, so that $A : a :: \frac{1}{B} : \frac{1}{b} :: b : B$.

For example, if the area of a triangle be given the base varies inversely as the altitude, for if A and a be the altitudes and B and b the bases of two equal triangles, then $AB = ab$ $\therefore A : a :: b : B$ or $A : a :: \frac{1}{B} : \frac{1}{b}$ or $A \propto \frac{1}{B}$

233. One quantity is said to *vary as two others jointly*, if when the first is changed in any manner the *product* of the other two is changed in the same proportion.

That is $A \propto BC$ (A varies as B and C jointly) when if A be changed to a the product BC must be changed to bc in such a way that $A : a :: BC : bc$.

Thus, the area of a triangle varies as the base and altitude jointly; for if A, B *and* P represent the area, base and altitude of any triangle, and a, b, p the area, base and altitude of any other triangle, then $A = \frac{1}{2} BP$ and $a = \frac{1}{2} bp$ $\therefore \frac{A}{a} = \frac{BP}{bp}$ $\therefore A : a :: BP : bp$ $\therefore A \propto BP$.

234. One quantity is said to *vary directly as a second* and *inversely as a third*, when the first cannot be changed in any manner, but the *quotient* of the second by the third is changed in the same proportion.

That is $A \propto \frac{B}{C}$ (A varies directly as B and inversely as C), when, if A be changed to a, $\frac{B}{C}$ must be changed to $\frac{b}{c}$ so that $A : a :: \frac{B}{C} : \frac{b}{c}$

Thus, the base of a triangle varies directly as the area and inversely as the altitude; for taking A, B, P; a, b and p as in last article $\frac{BP}{bp} = \frac{A}{a}$. multiplying both $\frac{p}{P}$ we get $\frac{B}{b} = \frac{Ap}{aP} = \frac{A}{P} \div \frac{a}{p}$ $\therefore$ $B : b :: \frac{A}{P} : \frac{a}{p}$ or $B \propto \frac{A}{P}$

THEOREMS.

235. THEOREM 1.—*If one quantity vary as another, it is equal to some constant multiple of that other. That is, if* A $\propto$ B *then* A = mB *where* m *is a constant quantity.*

DEMONSTRATION.—For if $A \propto B$ then $A : a :: B : b$, alternately $A : B :: a : b$ $\therefore$ $\frac{A}{B} = \frac{a}{b}$, let $\frac{a}{b} = m$, then $\frac{A}{B} = m$ $\therefore$ $A = mB$ where m is a constant quantity.

NOTE 1.—This principle enables us to convert a variation into an equation and is therefore made use of in almost every problem and theorem in variation.

NOTE 2.—Hence if m is a constant quantity and $A = mB$ then $A \propto B$, i. e A varies as B; also if $A = \frac{m}{B}$ then $A \propto \frac{1}{B}$ i.e. A varies inversely as B; also if $A = \frac{mB}{C}$ then $A \propto \frac{B}{C}$ i. e. varies directly as B and inversely as C.

Also, if $A = mBC$, then $A \propto BC$ i. e. A varies as B and C jointly.

236. THEOREM III.—*If $A \propto B$ and $B \propto C$, then $A \propto C$.*

DEMONSTRATION.—By Theorem I, $A = mB$ and $B = nC$ where m and n are constants, then $A = mnC$, that is $A \propto C$, because both m and n being constant, mn their product is also constant.

NOTE.—Also if $A \propto B$ and $B \propto \frac{1}{C}$ then $A \propto \frac{1}{C}$.

237. THEOREM III.—*If $A \propto C$ and $B \propto C$ then $A \pm B \propto C$ and $\sqrt{(AB)} \propto C$.*

DEMONSTRATION.—By Theorem I, $A = mC$ and $B = nC$ where m and n are constants. Then $A \pm B = m\,C \pm n\,C = (m \pm n)\ C$ $\therefore$ $A \pm B \propto C$, because $m \pm n$ is a constant quantity.

Also $\sqrt{(AB)} = \sqrt{(mC \times nC)} = \sqrt{(mnC^2)} = \sqrt{(mn)}C$ $\therefore$ $\sqrt{AB} \propto C$.

238. THEOREM. IV.—*If $A \propto BC$, then $B \propto \frac{A}{C}$ and $C \propto \frac{A}{B}$.*

DEMONSTRATION.—By Theorem I, $A = mBC$, then $B = \frac{A}{mC} = \frac{1}{m} \cdot \frac{A}{C}$ $\therefore B \propto \frac{A}{C}$ and $C = \frac{A}{mB} = \frac{1}{m} \cdot \frac{A}{B}$ $\therefore C \propto \frac{A}{B}$.

239. THEOREM V.—*If $A \propto B$ and $C \propto D$, then $AC \propto BD$.*

DEMONSTRATION.—By Theorem I, $A = mB$ and $C = nD$, then $AC = mnBD$ and $\therefore AC \propto BD$.

240. THEOREM VI.—*If $A \propto B$ then $A^n \propto B^n$.*

DEMONSTRATION.—By Theorem I, $A = mB$, then $A^n = m^n B^n$, but m is a constant quantity $\therefore A^n \propto B^n$.

NOTE.—So also if $A \propto B$ then $\sqrt[n]{A} \propto \sqrt[n]{B}$.

241. THEOREM VII.—*If $A \propto B$ and P be any other quantity then $AP \propto BP$ and $\frac{A}{P} \propto \frac{B}{P}$.*

DEMONSTRATION.—By Theorem I, $A = mB$ hence $PA = mPB$ $\therefore PA \propto PB$.

Also $A = mB$ $\therefore \frac{A}{P} = \frac{mB}{P} = m\frac{B}{P}$ $\therefore \frac{A}{P} \propto \frac{B}{P}$

NOTE.—Hence $\frac{A}{B}$ is constant, for if $A \propto B$ dividing both by B, we have $\frac{A}{B} \propto \frac{B}{B} \propto 1$.

242. THEOREM VIII.—*When three quantities are so related that the increase or decrease of one depends upon the increase or decrease of the other two, in such a way that if either of these latter be invariable the first varies as the other, then when both vary the first varies as their product. That is, if $A \propto B$ when C is constant and $A \propto C$ when B is constant, then $A \propto BC$ when both B and C are variable.*

DEMONSTRATION.—The variations of A depends upon the variations of two other quantities B and C; let the variations of these take place separately, and when B is changed to b let A be changed to a, and when C is changed to c let a be changed to a'. Then

$A : a :: B : b$; and

$a : a' :: C : c$ and by compounding these we have

$A : a' :: BC : bc$ $\therefore$ (Art. 229) $A \propto BC$.

NOTE.—In a similar way it may be shown that when there is any number of quantities, A, B, C, D &c., such that A varies as each of the others when the rest are constant—then, when they are all changed, A varies as their product.

Ex. 1. If $x \propto yz^2$ and 2, 3 and 5 be contemporaneous values of x, y and z, express x in terms of yz.

OPERATION.

Since $x \propto yz^2$ $\therefore x = myz^2$ and when $x = 2$, $y = 3$ and $z = 5$, then substituting these values we have $2 = m \times 3 \times 5^2 = 75m$ $\therefore m = \frac{2}{75}$. Then $x = myz^2$ or $x = \frac{2}{75}yz^2$.

Ex. 2. Given that $a \propto b$ and that when $a = 2$, $b = 1$, find the value of a when $b = 5$.

OPERATION.

Since $a \propto b$ $\therefore$ $a = mb$ or $2 = m$, because $a = 2$ and $b = 1$. Then when $b = 5$ we have $a = mb = 2 \times 5 = 10$.

Ex. 3. Given that $x \propto yz$, and that $x = 2$ when $y = z = 2$, find the value of x when $y = z = 3$.

OPERATION.

Since $x \propto yz$ $\therefore$ $x = myz$, that is $2 = m \times 2 \times 2 = 4m$ $\therefore$ $m = \frac{1}{2}$
Then $x = myz = \frac{1}{2} \times 3 \times 3 = \frac{9}{2} = 4\frac{1}{2}$ when $y = z = 3$.

Ex. 4. If $4y + 3z \propto 5y + 4z$, shew that $y \propto z$.

OPERATION.

$4y + 3z \propto 5y + 4z$ or $4y + 3z = m(5y + 4z) = 5my + 4mz$
$\therefore 4y - 5my = 4mz - 3z$ or $(4 - 5m)y = (4m - 3)z$ or $y = \left(\frac{4m-3}{4-5m}\right)z$
or $y = z$ multiplied by the constant quantity $\frac{4m-3}{4-5m}$ $\therefore$ $y \propto z$.

Ex. 5. If y = the sum of three quantities of which the first $\propto x^2$, the second $\propto x$, and the third is constant, and when $x = 1, 2, 3$, $y = 6, 11, 18$ respectively, express y in terms of x.

OPERATION.

The first quantity $\propto x^2$ and is $\therefore = mx^2$, similarly the second quantity $\propto x$ and is therefore $= nx$, and the third quantity is constant, and is $\therefore = p$, say. Then y being = the sum of these we have $y = mx^2 + nx + p$, and taking $x = 1, 2, 3$ and $y = 6, 11, 18$, we get the three equations :—

$$\left.\begin{aligned} 6 &= m + n + p \\ 11 &= 4m + 2n + p \\ 18 &= 9m + 3n + p \end{aligned}\right\}$$

which when solved give $m = 1$; $n = 2$, and $p = 3$, and substituting these in the equation $y = mx^2 + nx + p$ we have $y = x^2 + 2x + 3$.

EXERCISE LVIII.

1. If $mx^2 + y \propto cx^2 - dy$ shew that $x \propto y$.

2. Given that $x \propto y$ and that when $x = 7$, $y = 3$ find the equation between x and y.

3. Given that $x =$ the sum of two quantities whereof one is constant and the other varies inversely as y, and when $y = 3$, $x = 1$ when $y = 1$, $x = 2$, find the value of x when $y = 15$.

4. Given that $x^2 \propto y^3$ and $x = 2$ when $y = 4$ find the equation between x and y.

5. If $x =$ the sum of two quantities whereof one is constant and the other $\propto xy$, and when $x = 2$, $y = 3$, when $x = 3$, $y = -3$, express x in terms of y.

6. If $y =$ the sum of three quantities, of which the first is constant, the second $\propto x$, and the third $\propto x^2$; and when $x = 3$, 5, 7, $y = 0$, -12 -32 respectively; find the equation between x and y.

7. Given that $y =$ the sum of two quantities one of which varies as the square of x, while the other varies as x inversely, and that when $x = 5$, $y = 7$ and when $x = 9$, $y = 5$ find the equation between x and y.

8. Given that $y \propto (b^2 + x^2)$, and when $x = \sqrt{(a^2 - b^2)}$, $y = \frac{a^2}{b}$ find the equation between x and y.

9. If x, y, z be all variable quantities such that $z - x - y$ is constant, and $(x + y + z)(x - y - z) \propto yz$, prove that $x - y + z \propto yz$.

10. A locomotive engine without a train, can go 24 miles per hour, and its speed is diminished by a quantity which varies as the square root of the number of cars attached. With 4 cars its speed is 20 miles per hour. Find the greatest number of cars the engine can move.

SECTION XI.

PROGRESSIONS, PERMUTATIONS, AND COMBINATIONS.

ARITHMETICAL PROGRESSION.

243. Quantities are said to be in Arithmetical Progression when they increase or decrease by a *common difference*.

Thus, 4, 6, 8, 10, 12, &c., are in arithmetical progression, the common difference being 2.

$21a$, $18a$, $15a$, $12a$, $9a$, $6a$, &c., are in arithmetical progres., the common difference being $-3a$.

$3a + 5a + 7a + 9a$, &c., are in arith. progress., the common difference being $2a$.

244. In every progression the first and last terms are called the *extremes*, and the intermediate terms the *means*.

245. In *arithmetical progression* there are five things to be considered:

1. *The first term.*
2. *The last term.*
3. *The common difference.*
4. *The number of terms.*
5. *The sum of the series.*

These quantities are so related to one another that any three of them being given, the other two can be found, and hence there are 20 distinct cases arising from these combinations.

246. If we represent these five quantities by letters, thus,

a = *the first term*, l = *the last term*, d = *the common difference*,
n = *the number of terms*, s = *the sum of the series*,

the general expression for an arithmetical series will become

$$a+(a+d)+(a+2d)+(a+3d)+(a+4d)+(a+5d)+, \&c.,$$

where the coefficient of d is always *one less than the number of the term*. Thus, in the third term the coefficient of d is 2, which is 1 less than the number of the term; in the *fifth* term the coefficient of d is 4, which is 1 less than the number of the term, &c.

Hence $l = a + (n-1)d$; that is, the *last term* of an arithmetical series is equal to the *first term* added to the product of the *common difference* by *one less than the number of terms.*

247. Since the sum of the series is equal to the sum of all the terms taken in any order whatever, we have

$$\begin{array}{rl|l|l|l|l|l|l|l}
 & s = a + & a+d+ & a+2d+ & a+3d+ & \ldots l-3d+ & l-2d+ & l-d+ & l \\
\text{Also} & s = l + & l-d+ & l-2d+ & l-3d+ & \ldots a+3d+ & a+2d+ & a+d+ & a
\end{array}$$

Hence $2s = (a+l)+(a+l)+(a+l)+(a+l)+ \ldots\ldots$ to n terms.
But $(a+l)+(a+l)\ldots\ldots$ to n terms $= (a+l)n$.

Therefore $2s = (a+l)n$, and dividing these equals by 2, we have $s = (a+l)\frac{n}{2}$. That is, *the sum* of the series is found by adding together the *first* and *last terms*, and multiplying *their sum* by *half the number of terms.*

248. From the formula obtained in Art. 247, we find by transposing the terms

$$l = a + (n-1)d \qquad\qquad d = \frac{l-a}{n-1}$$

$$a = l - (n-1) \qquad\qquad dn = \frac{l-a}{d} + 1$$

and substituting these values of l, a, d, and n in the formula obtained in Art. 247, we find

$$s = \{2a + (n-1)d\}\frac{n}{2}$$

$$s = \{2l - (n-1)d\}\frac{n}{2}$$

$$s = \frac{(l-a)(l+a)}{2d} + \frac{l+a}{2}.$$

We thus obtain the five fundamental formulas from which the other fifteen are derived, by transposing the terms, &c. Thus,

$l = a + (n-1)d$	gives	formulas for	l, a, n, d	$= 4$
$s = (a + l)\frac{n}{2}$	"	"	s, a, l, n	$= 4$
$s = \{2a + (n-1)d\}\frac{n}{2}$		"	s, a, n, d	$= 4$
$s = \{2l - (n-1)d\}\frac{n}{2}$		"	s, l, n, d	$= 4$
$s = \frac{(l+a)(l-a)}{2d} + \frac{l+a}{2}$		"	s, a, l, d	$= 4$
			Total	20

249. By means of these equations when any three of the quantities a, d, l, n, s, are given, we may find a fourth, and may moreover proceed to the solution of many problems which without their aid would be difficult or even impossible. The student is recommended to carefully study the following examples :—

Ex. 1. Find the sum of the first 50 terms of the series $4a + 6a + 8a + 10a + \&c.$

OPERATION.

$$s = \{2a + (n-1)d\}\frac{n}{2} = \{8a + (50-1)2a\}\tfrac{50}{2} = (8a + 49 \times 2a)25$$
$$= (8a + 98a)25 = 106a \times 25 = 2650a.$$

Ex. 2. Given 3, the first term, and 55, the last term, of a series consisting of 27 terms, to find the common difference.

OPERATION.

$$l = a + (n-1)d \text{ or } (n-1)d = l - a \quad \therefore d = \frac{l-a}{n-1}$$

$$d = \frac{55-3}{27-1} = \frac{52}{26} = 2.$$

Ex. 3. Insert 5 arithmetical means between 1 and 23.

OPERATION.

Since there are five means and two extremes, there are in all 7 terms, and we must find the common difference of an arithmetical series of 7 terms whose first term is 1 and last term 23.

$$d = \frac{l - a}{n - 1} = \frac{23 - 1}{7 - 1} = \frac{22}{6} = 3\tfrac{2}{3}.$$

Hence the series is 1, $4\frac{2}{3}$, $8\frac{1}{3}$, 12, $15\frac{2}{3}$, $19\frac{1}{3}$, 23.

Ex. 4. How many terms of the series $6 + 8\frac{1}{3} + 10\frac{2}{3}$, &c., make up 3795?

OPERATION.

$$s = \{2a + (n - 1)d\}\frac{n}{2};\ 3795 = \{12 + (n - 1)2\tfrac{1}{3}\}\frac{n}{2}$$

$$7590 = 12n + (n^2 - n)2\tfrac{1}{3};\ 22770 = 36n + 7n^2 - 7n;\ 7n^2 + 29n = 22770$$

$$n^2 + \tfrac{29}{7}n + (\tfrac{29}{14})^2 = \tfrac{22770}{7} + \tfrac{841}{196} = \tfrac{638401}{196};\ n + \tfrac{29}{14} = \pm\tfrac{799}{14}$$

$$n = \frac{\pm 799 - 29}{14} = \tfrac{770}{14} = 55.$$

NOTE.—The negative value $-57\frac{5}{7}$ does not satisfy the conditions of the question, and is therefore inadmissible.

Ex. 5. The sum of four numbers in arithmetical progression is 32, and the sum of their squares is 276. Required the numbers.

OPERATION.

Let x = the second number and y = the com. diff.

Then $x - y$, x, $x + y$, and $x + 2y$ is the series.

$\therefore x - y + x + x + y + x + 2y = 4x + 2y = 32$ or $2x + y = 16$.

Also $(x - y)^2 + x^2 + (x + y)^2 + (x + 2y)^2 = 4xy + 4x^2 + 6y^2 = 276$ or $2x^2 + 2xy + 3y^2 = 138$.

And $y = 16 - 2x$ $\therefore 2x^2 + 2x(16 - 2x) + 3(16 - 2x)^2 = 138$.

That is, $2x^2 + 32x - 4x^2 + 768 - 192x + 12x^2 = 138$.

That is, $10x^2 - 160x = -630$; $x^2 - 16x = -63$; $x^2 - 16x + 64 = 1$.

$x - 8 = \pm 1$ or $x = 9$ or 7.

$y = 16 - 2x = 16 - 18 = -2$, or $16 - 14 = 2$.

Hence taking $x = 9$ and $y = -2$ we have the series 11, 9, 7, 5; taking $x = 7$ and $y = 2$ we have 5, 7, 9, 11.

Otherwise, let $x - 3y$, $x - y$, $x + y$, and $x + 3y$ represent the number, where $2y$ = the common difference.

Then $x - 3y + x - y + x + y + x + 3y = 4x = 32 \therefore x = 8$.

$(x - 3y)^2 + (x - y)^2 + (x + y)^2 + (x + 3y)^2 = 4x^2 + 20y^2 = 276$ or $20y^2 = 276 - 256 = 20$.

$y^2 = 1$, $y = \pm 1$. Hence $x - 3y = 8 \mp 3 = 5$ or 11, &c.

EXERCISE LIX.

Sum the following series:

1. 63, 65, 67, &c., to 31 terms and also to n terms.
2. $-200, -188, -176, -164, -$ &c., to 22 terms and to n terms.
3. $2, 3\frac{1}{2}, 5$, &c., to 17 terms and also to $2m + p$ terms.
4. $\frac{2}{3}, 0, -\frac{2}{3}, -1\frac{1}{3}$, &c., to 11 terms.

Find the 17th and 28th and nth terms of the series:

5. 2, 5, 8, &c.
6. $3, -2, -7$, &c.
7. $2\frac{1}{2}, 3\frac{3}{14}, 3\frac{13}{14}$, &c.
8. Insert 3 arithmetical means between 3 and 33.
9. Insert 4 arithmetical means between 9 and -66.
10. Insert 7 arithmetical means between -1 and 100.
11. Find the sum of 73 terms of the series 1, 2, 3, 4, &c.
12. What is the nth term of the series, 1, 3, 5, 7, &c.
13. Prove that the sum of n terms of the series 1, 3, 5, 7, &c., is equal to n^2.
14. If a body falling to the earth descends a feet the first second, $3a$ feet the second, $5a$ feet the third, and so on; how far will it fall in t seconds?
15. How far will the body (Question 14) fall during the 20th second and during the t th second.
16. There are four numbers in arithmetical progression, of which the sum of the squares of the extremes is 200, and the sum of the squares of the means is 136. Find the numbers.
17. There are four numbers in arithmetical progression whose continued product is 1680 and common difference 4. What are the numbers?
18. There are five numbers in arithmetical progression whose sum is 25 and continued product 945. What are the numbers?

19. A man borrowed \$60 at 6 per cent. simple interest, per year of 360 days. How much must he pay daily to cancel the debt, principal, and interest, in 60 days?

20. Prove that the sum of n terms of the natural numbers 1, 2, 3, &c., is $\frac{n(n+1)}{2}$.

21. Prove that the sum of the squares of the first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.

22. How many terms of the series 2, 11, 20, &c., are required to make up 517?

23. Find the arithmetical series the last three terms of which amount to 96, and the preceding four terms of which added together make up 86.

24. Find the arithmetical series of which the 5th and 7th terms are respectively 7 and 5.

25. Given s the sum of an arithmetical series $= bn + cn^2$ for all values of n, find the t th term of the series.

26. Prove that the sum of the $(m - n)$th and $(m + n)$th terms of an arithmetical series is double the mth term.

27. In an arithmetical progression if the $(p + q)$th term $= m$, and the $(p - q)$th term $= n$, prove that the qth term of the series is $= m - (m - n)\frac{p}{2q}$.

28. Sum to n terms the arithmetical progression whose pth term is $7 - \frac{p}{2}$.

29. There are three numbers in arithmetical progression, such that the square of the first added to the product of the other two is 16; the square of the second added to the product of the other two is 14. What are the numbers?

30. The sum of four whole numbers in arithmetical progression is 20, and the sum of their reciprocals is $\frac{25}{24}$. Required the numbers.

GEOMETRICAL PROGRESSION.

250. Quantities are said to be in geometrical progression when they increase or decrease by a common multiplier.

Thus, 2, 4, 8, 16, 32, &c., are in geometrical progression, the common multiplier being 2.

$5a, -15a^2, 45a^3, -135a^4$, &c., are in geometrical progression the common multiplier being $-3a$.

251. In *geometrical progression* there are five things to be considered:

1. *The first term.*
2. *The last term.*
3. *The common ratio.*
4. *The number of terms.*
5. *The sum of the series.*

As in arithmetical progression, these five quantities are so related that any three of them being given the other two can be found, and hence there are 20 distinct cases arising from their combination.

252. Representing these five quantities by letters, thus,

a = *the first term*, l = *the last term*, r = *the common ratio*,
n = *the number of terms*, s = *the sum of the series*,

the general expression for a geometrical series becomes

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \&c.,$$

where the index of r is always *one less* than the number of the term.

Thus, in the third term the index of r is 2, which is *one less* than the number of the term; in the fifth term the index of r is 4, which is *one less* than the number of the term, &c.

Hence $l = ar^{n-1}$; that is, the last term is equal to the first term multiplied by the common ratio raised to that power which is indicated by one less than the number of terms.

253. Since the sum of the series is equal to the sum of all the terms,

$s = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$, multiplying by r, we get
$sr = ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n$.

Hence $sr - s = ar^n - a$; or $s(r-1) = a(r^n - 1)$, and therefore
$$s = \frac{a(r^n - 1)}{r - 1}.$$

254. From the formula obtained in Art. 252 we get by transposing the terms, &c.,

$$l = ar^{n-1} \qquad r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$$

$$a = \frac{l}{r^{n-1}} \qquad n = \frac{log.\, l - log.\, a}{log.\, r} + 1$$

And substituting these values of l, a, r, n, in the formula obtained in Art. 254, we find

$$s = \frac{rl - a}{r - 1} \qquad s = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$$

$$s = \frac{l(r^n - 1)}{(r-1)r^{n-1}}$$

and these together with the two formulas obtained in Arts. 252 and 253,

$$s = \frac{a(r^n - 1)}{r - 1}$$

$$l = ar^{n-1}$$

are the fundamental formulas of geometrical progression from which the other fifteen are derived by reduction. Thus,

$s = \dfrac{rl - a}{r - 1}$	gives formulas for	s, r, l, and a,	$= 4$
$s = \dfrac{l(r^n - 1)}{(r - 1)r^{n-1}}$	"	s, r, l, and n,	$= 4$
$s = \dfrac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$	"	s, l, n, and a,	$= 4$
$s = \dfrac{a(r^n - 1)}{r - 1}$	"	s, r, a, and n,	$= 4$
$l = ar^{n-1}$	"	l, a, r, and n,	$= 4$
		Total	20

255. When the common ratio of a geometrical series is a proper fraction, the series is a descending one, and if the number of terms is infinitely great, r^n becomes infinitely small; i. e., r^n becomes $= 0$; hence ar^n in formula $\frac{ar^n - a}{r - 1}$ becomes equal to zero, and the formula for finding the sum becomes $\frac{-a}{r-1} = \frac{a}{1-r}$. The expression $\frac{a}{1-r}$ properly speaking, however, represents the *limit* of the sum of the infinite series rather than the sum itself.

256. By means of these formulas many problems in geometrical progression may be solved, but as a rule questions in which the value of n is sought are incapable of solution except by the higher analysis.

Ex. 1. Find the last term and the sum of the series 3, 6, 12, &c., to 11 terms.

OPERATION.

$$l = ar^{n-1} = 3 \times 2^{10} = 3 \times 1024 = 3072$$

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{3(2^{11} - 1)}{2 - 1} = 3(2048 - 1) = 3 \times 2047 = 6141.$$

Ex. 2. Find the limit to the sum of the series 8 + 4 + 2 + 1 + &c., ad infinitum.

OPERATION.

$$s = \frac{a}{1 - r} = \frac{8}{1 - \frac{1}{2}} = \frac{8}{\frac{1}{2}} = 16.$$

Ex. 3. Find the 7th term and the sum of 8 terms of the series $\frac{5}{6}, \frac{5}{9}, \frac{10}{27}$.

OPERATION.

The common ratio is always = 2nd term ÷ 1st term.

Hence in this question $r = \frac{5}{9} \div \frac{5}{6} = \frac{6}{9} = \frac{2}{3}$

$$l = ar^{n-1} = (\tfrac{5}{6})(\tfrac{2}{3})^6 = \tfrac{5}{6} \times \tfrac{64}{729} = \tfrac{160}{2187}$$

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{\frac{5}{6}((\frac{2}{3})^7 - 1)}{\frac{2}{3} - 1} = \frac{\frac{5}{6}(\frac{128}{2187} - 1)}{-\frac{1}{3}} = \frac{\frac{5}{6}(\frac{2187 - 128}{2187})}{\frac{1}{3}}$$

$$= \tfrac{5}{2}(\tfrac{2059}{2187}) = \tfrac{10295}{4374} = 2\tfrac{1547}{2187}.$$

Ex. 4. Insert three geometrical means between 4 and 324.

OPERATION.

$l = ar^{n-1} \ \therefore \ r^{n-1} = \dfrac{l}{a}.$

And since there are here 3 means and 2 extremes there are in all 5 terms, then $r^{5-1} = \frac{324}{4}$, $r^4 = 81$, whence r is evidently $= 3$, and the series is 4, 12, 36, 108, 324.

Ex. 5. Find six numbers in geometrical progression such that the sum of the extremes is 99, and the sum of the other four terms, 90.

OPERATION.

The sum of the six terms is evidently $99 + 90 = 189$.

Let x = the first term and y = the common ratio.

Then $x, xy, xy^2, xy^3, xy^4, xy^5$, represent the terms

$$s = 189 = \frac{lr - a}{r - 1} = \frac{xy^6 - x}{y - 1} = \frac{x(y^6 - 1)}{y - 1}$$

$$\therefore x = \frac{189(y-1)}{y^6 - 1}. \quad \text{But } xy^5 + x = x(y^5 + 1) = 99 \ \therefore x = \frac{99}{y^5 + 1}$$

$$\therefore \frac{189(y-1)}{y^6-1} = \frac{99}{y^5+1}; \quad \frac{21(y^2-1)}{y^6-1} = \frac{11}{y^4 - y^3 + y^2 - y + 1}$$

$$\therefore \frac{21}{y^4+y^2+1} = \frac{11}{y^4 - y^3 + y^2 - y + 1}$$

$$\therefore 21y^4 - 21y^3 + 21y^2 - 21y + 21 = 11y^4 + 11y^2 + 11$$

$$10y^4 + 10y^2 + 10 = 21y^3 + 21y$$

$$10(y^4 + y^2 + 1) = 21y(y^2 + 1)$$

$$10(y^4 + 2y^2 + 1 - y^2) = 21y(y^2 + 1)$$

$$10(y^2 + 1)^2 - 10y^2 = 21y(y^2 + 1)$$

$$10(y^2 + 1)^2 - 21y(y^2 + 1) = 10y^2$$

$$(y^2 + 1)^2 - \frac{21y}{10}(y^2+1) + \left(\frac{21y}{20}\right)^2 = \frac{441y^2}{400} + \frac{400y^2}{400} = \frac{841y^2}{400}$$

$$y^2 + 1 - \frac{21y}{20} = \pm\frac{29y}{20}$$

$$y^2 + 1 = \frac{21y \pm 29y}{20} = \frac{50y}{20} = \frac{5y}{2}$$

$$2y^2 - 5y = -2; \ 16y^2 - 40y + 25 = -16 + 25 = 9$$

$$4y - 5 = \pm 3; \ 4y = 5 + 3 = 8 \ \therefore y = 2$$

$$x = \frac{99}{y^5 + 1} = \frac{99}{33} = 3.$$

Therefore the series is 3, 6, 12, 24, 48, 96.

Ex. 6. The sum of four numbers in geometrical progression is equal to the common ratio + 1, and the first term is $\frac{1}{17}$. Required the numbers.

OPERATION.

Let r = the common ratio.

Then the numbers are $\frac{1}{17}$, $\frac{r}{17}$, $\frac{r^2}{17}$, and $\frac{r^3}{17}$.

$$\text{Then } 1 + r = \frac{1 + r + r^2 + r^3}{17} = \frac{1 + r + r^2(1 + r)}{17} = \frac{(1+r)(1+r^2)}{17}$$

$$\therefore 1 = \frac{1 + r^2}{17} \text{ or } r^2 + 1 = 17 \text{ ; } r^2 = 16, \therefore r = \pm 4,$$

and the numbers are $\frac{1}{17}$, $\frac{4}{17}$, $\frac{16}{17}$, $\frac{64}{17}$,
or $\frac{1}{17}$, $-\frac{4}{17}$, $\frac{16}{17}$, $-\frac{64}{17}$.

EXERCISE LX.

Find the last term and the sum of:

1. $3 + 9 + 27 +$ &c. to 6 terms.
2. $1 + 2 + 4 +$ &c. to 9 terms.
3. $\frac{2}{7} + \frac{4}{7} + \frac{8}{7} +$ &c. to 7 terms.
4. $3 - 6 + 12 -$ &c. to 12 terms.
5. $4 - 5 + 6\frac{1}{4} -$ &c. to 6 terms.
6. $30 - 15 + 7\frac{1}{2} -$ &c. to 8 terms.

Find the limit to the sum of the infinite series:

7. $-1\frac{1}{3} + \frac{8}{9} - \frac{16}{27} +$ &c.
8. $\frac{2}{5} + \frac{4}{15} + \frac{8}{45} +$ &c.
9. $7 - 3\frac{1}{2} + 1\frac{3}{4} -$ &c.
10. $64 - 32 + 16 -$ &c.
11. $\cdot\dot{6}2\dot{3}$.
12. $\cdot\dot{7}$.
13. $\cdot9\dot{7}\dot{6}$.
14. $\cdot862\dot{3}\dot{2}$.

Sum the following series:

15. $1 + 3 + 9 +$ &c. to n terms.
16. $2 - \frac{4}{5} + \frac{8}{25} -$ &c. to n terms.
17. $2 + \sqrt{8} + 4 +$ &c. to 10 terms.
18. $a^p + a^{p+q} + a^{p+2q} +$ &c. to n terms.
19. Insert three geometrical means between 1 and $\frac{16}{81}$.
20. Insert seven geometrical means between 2 and 13122.
21. Insert three geometrical means between 9 and $\frac{1}{9}$.
22. The sum of the first and third of four numbers in G. P. is 148, and the sum of the second and fourth is 888. What are the numbers?

23. The sum of the first and second of four numbers in G. P. is 15, and the sum of the third and fourth is 60. Required the numbers.

24. The sum of \$315 was divided among three persons in such a way that the first received \$135 more than the last. The three shares being in G. P., required what they were. Interpret the negative result obtained in the solution.

25. There are five whole numbers, the first three of which are in G. P.; the last three in A. P.; the second number being the common difference of these three terms. The sum of the last four is 40, and the product of the second and last is 64. Required the numbers.

26. Prove that the sum of n terms of the series $a + (a+b)r + (a+2b)r^2 + (a+3b)r^3 + \&c.$,

$$= \frac{a - \{a + (n-1)b\}r^n}{1-r} + \frac{br(1-r^{n-1})}{(1-r)^2}$$

27. If a, b, c, d, are four quantities in G. P., prove that $a^2 + b^2 + c^2 > (a - b + c)^2$, and that $(a+b+c+d)^2 = (a+b)^2 + (c+d)^2 + 2(b+c)^2$.

28. In a G. P. if the $(p+q)$th term $= m$, and the $(p-q)$th term $= n$, show that the pth term $= \sqrt{mn}$, and also that the qth term $= m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$.

29. The sum of three numbers in G. P. is 35, and the mean term is to the difference of the extremes as 2 : 3. Required the numbers.

30. There is a number consisting of three digits, the first of which is to the second as the second is to the third; the number itself is to the sum of its digits as 124 : 7, and if 594 be added to it, its digits will be inverted. Required the number.

HARMONICAL PROGRESSION.

257. Quantities are said to be in harmonical progression when their reciprocals are in arithmetical progression, or when of any three consecutive terms the first is to the third

as the difference between the first and second is to the difference between the second and third.

Thus, a, b, and c are said to be in H. P. when $a : c :: a - b : b - c$. Also, since 3, 7, 11, &c., are A. P., their reciprocals $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{11}$, &c., are in H. P.

258. It may be easily proved that the reciprocals of a series of quantities in H. P. are in A. P., as follows:—

Let a, b, c be in H. P. Then $a : c :: a - b : b - c$ or $a(b - c) = c(a - b)$, or $ab - ac = ac - bc$, and dividing each of these by abc we have $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$. But when the difference between the first and second is the same as the difference between the second and third, the three quantities are said to be in A. P.

259. No general rule can be given for finding the sum of a series of terms in H. P., but, by inverting the given terms so as to form a series in A. P., many useful problems may be solved.

Ex. 1. Continue the H. series $2\frac{1}{2}$, $1\frac{2}{3}$, $1\frac{1}{4}$, three terms each way.

OPERATION.

Since $\frac{5}{2}$, $\frac{5}{3}$, $\frac{5}{4}$, are in H. P., their reciprocals, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, are in A. P., and their common difference $= \frac{1}{5}$. Hence $-\frac{1}{5}$, $\frac{0}{5}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{5}$, $\frac{6}{5}$, $\frac{7}{5}$, is the continued A. series, and these terms inverted give us for the required H. series -5, ∞, 5, $2\frac{1}{2}$, $1\frac{2}{3}$, $1\frac{1}{4}$, 1, $\frac{5}{6}$, $\frac{5}{7}$.

NOTE.—The second term of the A. P. is $\frac{0}{5}$, which inverted gives us $\frac{5}{0}$ which $= \infty$. (See Art. 66.)

Ex. 2. Insert four H. means between 2 and 6.

OPERATION.

Insert four A. means between $\frac{1}{2}$ and $\frac{1}{6}$. Here $d = \dfrac{\frac{1}{6} - \frac{1}{2}}{6 - 1} = -\dfrac{\frac{1}{3}}{5} = -\frac{1}{15}$. Hence the A series is $\frac{1}{2}$, $\frac{13}{30}$, $\frac{11}{30}$, $\frac{9}{30}$, $\frac{7}{30}$, $\frac{5}{30}$,

$\therefore$ H. series is 2, $2\frac{4}{13}$, $2\frac{8}{11}$, $3\frac{1}{3}$, $4\frac{2}{7}$, 6.

Ex. 3. Insert three H. means between 10 and 30.

OPERATION.

Insert 3 A. means between $\frac{1}{10}$ and $\frac{1}{30}$. Here $d = \dfrac{\frac{1}{30} - \frac{1}{10}}{5 - 1} = -\dfrac{\frac{1}{15}}{4} = -\frac{1}{60}$, and the A. series is $\frac{1}{10}$, $\frac{5}{60}$, $\frac{4}{60}$, $\frac{3}{60}$, $\frac{2}{60}$. Hence the H. series = 10, 12, 15, 20, 30.

Ex. 4. Find the nth term of the H. series $1\frac{1}{2}$, 1, $\frac{3}{4}$, &c.

OPERATION.

The nth term of the A. series $\frac{2}{3}$, 1, $\frac{4}{3}$, &c., $= a+(n-1)d = \frac{2}{3}+(n-1)\frac{1}{3}$ $= \frac{2}{3} + \frac{n}{3} - \frac{1}{3} = \frac{1}{3} + \frac{n}{3} = \frac{n+1}{3}$ $\therefore$ the nth term of the given H series is $\frac{3}{n+1}$.

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260. Let a and b be any two quantities, and let A be their arithmetical mean, G their geometrical mean, and H their harmonical mean. Then,

I. $A - a = b - A$ or $2A = a + b$ $\therefore$ $A = \frac{1}{2}(a + b)$. Art. 243.

II. $a : G :: G : b$ or $G^2 = ab$ $\therefore$ $G = \sqrt{ab}$. Arts. 224 and 250.

III. $a : b :: a - H : H - b$ or $aH + bH = 2ab$ $\therefore$ $H = \frac{2ab}{a+b}$. Art. 257.

261. *Hence the A. mean between two quantities is equal to half their sum, the G. mean between two quantities is equal to the square root of their product, and the H. mean between two quantities is equal to twice their product divided by their sum.*

262. THEOREM I.—*Taking* A, G, *and* H, *as in last article,* G *is the geometrical mean between* A *and* H.

DEMONSTRATION. Since $A = \frac{1}{2}(a + b)$ and $H = \frac{2ab}{a+b}$ $\therefore$ $AH = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab$, but $G^2 = ab$ $\therefore$ $G^2 = AH$. Extracting the square root of both, we have $G = \sqrt{AH}$, that is, G is the geometrical mean between A and H.

263. THEOREM II.—*Taking* A, G, *and* H *as in* Art. 260, *then of the three* A *is the greatest and* H *the least in magnitude.*

DEMONSTRATION. Because, (Art. 134) $a^2 + b^2 > 2\,ab$, $a^2 + 2ab + b^2 > 4ab$, and $a + b > \frac{4ab}{a+b}$, and $\frac{a+b}{2} > \frac{2ab}{a+b}$, but $\frac{a+b}{2} = A$, and $\frac{2ab}{a+b} = H$, $\therefore A > H$. And G being the geometrical mean between A and H is of intermediate magnitude, *i. e.*, is greater than H and less than A, $\therefore A > G > H$.

264. THEOREM III.—*Three quantities, a, b, c, are in A. P. or H. P., or G. P., according* $\frac{a-b}{b-c} = \frac{a}{a}$ *or* $\frac{a}{b}$ *or* $\frac{a}{c}$.

DEMONSTRATION I. $\frac{a-b}{b-c} = \frac{a}{a} = 1 \therefore a-b = b-c$ or $b = \frac{1}{2}(a+c)$.

II. $\frac{a-b}{b-c} = \frac{a}{b} \therefore ab - b^2 = ab - ac$ or $b^2 = ac \therefore b = \sqrt{ac}$.

III. $\frac{a-b}{b-c} = \frac{a}{c} \therefore a : c :: a-b : b-c$.

Ex. 5. Find the *A. G.* and *H.* means between $1\frac{1}{6}$ and 10.

OPERATION.

$A = \frac{1}{2}(a+b) = \frac{1}{2}(1\frac{1}{6} + 10) = \frac{1}{2} \times 11\frac{1}{6} = \frac{1}{2} \times \frac{67}{6} = \frac{67}{12} = 5\frac{7}{12}$.

$G = \sqrt{ab} = \sqrt{1\frac{1}{6} \times 10} = \sqrt{16} = 4$.

$H = \frac{2ab}{a+b} = \frac{2 \times 1\frac{1}{6} \times 10}{1\frac{1}{6} + 10} = \frac{32}{11\frac{1}{6}} = 2\frac{58}{67}$.

Ex. 6. The difference of the A. and H. means between two numbers is $1\frac{4}{5}$; find the numbers, one being four times as great as the other.

OPERATION.

$A = \frac{1}{2}(a+b)$ and $H = \frac{2ab}{a+b} \therefore A - H = \frac{a+b}{2} - \frac{2ab}{a+b}$

$= \frac{a^2 + 2ab + b^2 - 4ab}{2(a+b)} = \frac{(a-b)^2}{2(a+b)} = \frac{9}{5} \therefore$ since $a = 4b$ we have

$\frac{(4b-b)^2}{2(4b+b)} = \frac{(3b)^2}{2 \times 5b} = \frac{9b^2}{10b} = \frac{9b}{10} = \frac{9}{5} \therefore \frac{b}{10} = \frac{1}{5}$ or $b = 2$ and $a = 4b = 8$.

EXERCISE LXI.

1. Continue three terms each way the H. series, (I) $\frac{1}{7}, \frac{1}{5}, \frac{1}{3}$ (II) $\frac{1}{18}, \frac{1}{14}, \frac{1}{10}$; (III) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$; (IV) 14, $1\frac{5}{9}, \frac{14}{17}$; (V) $\frac{5}{11}, 1\frac{1}{4}, -1\frac{2}{3}$ (VI) $-\frac{1}{2}, \infty, \frac{1}{2}$.

2. Insert three H. means between 2 and 3; between 5 and 7 between 11 and 3; between $2\frac{1}{4}$ and $3\frac{1}{7}$; between 6 and $-\frac{2}{5}$.

3. Find the 5th, 11th, and nth terms of the H. series $2\frac{1}{2}$, 1, $\frac{5}{8}$.

4. Find the 6th, 10th, and last term of the H. series $4\frac{1}{3}, 6\frac{1}{2}, 13$.

5. Find the 4th and 8th terms of the H. series $\frac{1}{10}, \frac{1}{12}, \frac{1}{14}$.

6. Find the unknown terms of a H. series whose first term is 4 and fourth term 1.

7. Find the 8th term and the nth terms of a H. series whose first term is a and second term b.

8. Find the H. mean between $\dfrac{1}{m+n}$ and $\dfrac{1}{m-n}$.

9. Find the A. G. and H. means between 4 and 9.

10. Find the A. G. and H. means between 6 and $4\frac{1}{6}$.

11. If a, b, c, be three quantities in H. P., prove that $a^2 + c^2 > 2b^2$, if a and c are both positive or both negative.

12. If a, b, c, are in A. P., and a, mb, c, in G. P., prove that a, m^2b, c, are in H. P.

13. From each of three quantities in H. P. what quantity must be taken away in order that the three resulting quantities may be in G. P.?

14. The sum and difference of the A. and G. means between two quantities are 16 and 4 respectively. Required the numbers,

15. The A. mean between two numbers is $2\frac{2}{7}$ of the H. mean, and one of the numbers is 2. Required the other.

16. Find two numbers whose sum is 30 and H. mean $13\frac{1}{3}$.

17. Find two numbers whose difference is $16\frac{1}{4}$ and the G. mean between the H. and A. means of which is 9.

PERMUTATIONS, VARIATIONS, COMBINATIONS.

265. The *different orders* in which any given number of quantities can be arranged are called their *permutations* or *variations*.

Thus, the permutations of a, b, c, taken three together, are abc, acb, bac, bca, cab, cba; taken two together, they are ab, ba, ac, ca, bc, cb.

Note.—Some writers make a distinction between permutations and variations—limiting the application of the former term to those cases in which all the quantities are taken together, and calling others variations.

266. The *combinations* of any given number of things are the *different collections* that can be formed out of them without taking into consideration the order in which the quantities are placed.

Thus, the combinations that can be formed out of three things, a, b, c, are three in number, viz., ab, ac, and bc.

267. THEOREM I.—*The number of variations of* n *things taken* p *together is* $n(n-1)(n-2)\ldots.(n-p+1)$.

DEMONSTRATION. Let there be n different things a, b, c, d, &c. Then the number of variations which can be formed out of these n different things taken one at a time is manifestly $= n$.

From the n things a, b, c, d, &c., let us remove a, then there will remain $n-1$ things b, c, d, and the number of variations of these $n-1$ things taken singly will of course be $= n-1$. Now if we place a before each of these $n-1$ variations there will $n-1$ variations of a, b, c, d, &c., taken *two* and *two* together, in which a stands first. Similarly there will be $n-1$ such variations in which b stands first, and so of the rest. Therefore there are upon the whole $n(n-1)$ variations of n things taken *two* and *two* together.

Hence of $(n-1)$ things b, c, d, &c., taken two and two together, there are $(n-1)(n-2)$ variations, and placing a before each of these it appears there are $(n-1)(n-2)$ variations of n things a, b, c, &c., taken *three* and *three* together, in which a stands first, and as the same may be said of b, c, d, &c., there are upon the whole $n(n-1)(n-2)$ variations of n things taken three and three together.

Similarly the number of variations of n things taken four and four together, may be shown to be $n(n-1)(n-2)(n-3)$, and five and five together, $n(n-1)(n-2)(n-3)(n-4)$, and so on. Now it has been shown that variations of n things taken

2 together $= n(n-1)$. or $n(n-2+1)$

3 " $= n(n-1)(n-2)$ or $n(n-1)(n-3+1)$

4 " $= n(n-1)(n-2)(n-3)$ or $n(n-1)(n-2)(n-4+1)$

and so on. Hence the variations of n things taken p together $= n(n-1)(n-2)\ldots.(n-p+1)$.

Cor. 1. If $p = n$, that is, if the quantities are taken all together, the variations or permutations of n things is $n(n-1)(n-2)\ldots\ldots(n-n+1) = n(n-1)(n-2)\ldots\ldots3.2.1$, or, reversing the order of these terms we have permutations of n things $= 1.2.3.4\ldots\ldots n$.

Cor. 2. Hence denoting the variations of n things taken 1, 2, 3, 4, &c., p together by V_1, V_2, V_3, V_4, &c., V_p we have $V_1 = n$; $V_2 = n(n-1)$; $V_3 = n(n-1)(n-2)$; $V_4 = n(n-1)(n-2)(n-3)$; &c.; $V_p = n(n-1)(n-2)(n-3)\ldots\ldots(n-p+1)$.

NOTE.—For the sake of brevity $n(n-1)(n-2)\ldots\ldots3.2.1$ is frequently indicated by $\underline{|n}$ (read *factorial n.*) accordingly, $\underline{|n}$ denotes the continued product of the natural numbers from 1 to n inclusive.

268. THEOREM II.—*The number of permutations of* n *things taken all together, whereof* p *are* n's, q *are* b's, *and* r *are* c's, *is*

$$\frac{\underline{|n}}{\underline{|p}\ \underline{|q}\ \underline{|r}}.$$

DEMONSTRATION.—Let N denote the number of permutations under the given conditions. Then if we suppose that in any one of these N permutations we change the p a's into letters differing from all of the rest, we could from this single permutation produce $\underline{|p}$ different permutations, and as the same would be true for *each* of the N permutations, it appears that if the p a's are changed to letters differing from all the others, there will be $N\ \underline{|p}$ permutations of n letters, whereof there are still q b's and r c's.

If now the q b's were changed to letters differing from all the rest, it may be shown by similar reasoning that we should have $N\ \underline{|p}\ \underline{|q}$ variations of n things, whereof there still remain r c's.

Similarly, if the r c's are changed to letters differing from all the rest, we shall find that the number of permutations of n different things $= N\ \underline{|p}\ \underline{|q}\ \underline{|r}$. But the permutations of n *different* things is $\underline{|n}$.

Hence $N\ \underline{|p}\ \underline{|q}\ \underline{|r} = \underline{|n}$, and dividing both sides of the equation by $\underline{|p}\ \underline{|q}\ \underline{|r}$ we have $N = \dfrac{\underline{|n}}{\underline{|p}\ \underline{|q}\ \underline{|r}}$.

Ex. 1. How many variations can be made of 10 things taken 3, 5, 8, and 10 at a time?

OPERATION.

$V_3 = n(n-1)(n-2) = 10.9.8 = 720$

$V_5 = n(n-1)(n-2)(n-3)(n-4) = 10.9.8.7.6 = 30240$

$V_8 = n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)$
$= 10.9.8.7.6.5.4.3 = 1814400$

$V_{10} = 1.2.3.4\ldots.n = 1.2.3.4.5.6.7.8.9.10 = 3628800.$

Ex. 2. How many different words can be made with all the letters in the expression $a^4bc^2de^5$.

OPERATION.

We are to find the permutation of 13 letters, of which 4 are a's, 2 are c's, and 5 are e's.

$$N = \frac{|\underline{n}}{|\underline{p}\,|\underline{q}\,|\underline{r}} = \frac{1.2.3.4.5.6.7.8.9.10.11.12.13}{1.2.3.4 \times 1.2 \times 1.2.3.4.5}$$

$= 7 \times 9 \times 10 \times 11 \times 12 \times 13 = 1081080.$

Ex. 3. The number of variations of $n-2$ things 3 together : number of variations of n things 3 together :: 5 : 12. Find the value of n.

OPERATION.

$(n-2)(n-3)(n-4) : n(n-1)(n-2) :: 5 : 12$

$12(n-2)(n-3)(n-4) = 5n(n-1)(n-2)$

$12(n-3)(n-4) = 5n(n-1)$

$12(n^2-7n+12) = 5n^2-5n$

$12n^2-84n+144 = 5n^2-5n$ or $7n^2-79n = -144$

$196n^2-2212n+6241 = -4032+6241 = 2209$

$14n-79 = \pm 47 \therefore 14n = 126$, or $n = 9$.

Ex. 4. The variations of a certain number of things taken 3 together is 20 times as great as the number of variations of half as many things taken 3 together. Find the number of things.

OPERATION.

$n(n-1)(n-2) = 20 \times \frac{1}{2}n(\frac{1}{2}n-1)(\frac{1}{2}n-2)$

$$n(n-1)(n-2) = 10n\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right).$$

$n(n-1)(n-2) = \frac{5}{2}\,n(n-2)(n-4)$

and dividing both by $n(n-2)$ we have $n-1 = \frac{5}{2}(n-4)$ whence $n = 6$.

EXERCISE LXII.

1. In how many different ways can six different counters be arranged?

2. How many variations can be formed out of 8 things taken (I) 4 together, (II) 6 together, and (III) all together.

3. How many different words can be formed out of the expression $a^5b^4c^3d$?

4. Assuming that sixteen changes can be rung per minute, and that the bells are rung 10 hours each day, how long would it require to ring all the changes that can be rung on 12 bells?

5. If the number of permutations of n things 5 together is six times as great as the number 3 together, find n.

6. A landlord agrees to board a company of 10 persons as many days as they can sit in different positions at table, for \$5000. Assuming that the board of each is worth \$5 per week, how much does he lose by the transaction? What is his loss if the \$5000 is paid at once and placed at simple interest at 6 per cent. per annum till the close of the term of agreement?

7. The number of variations of 15 things taken n together is ten times as great as the number taken $(n-1)$ together. Find the value of n.

8. How many different words may be made of all the letters in the words *Constantinople*, *divisibility*, *octoroon*, *commemoration*.

9. How many different permutations can be formed with the letters in the words *algebra*, *demonstration*, *Toronto*.

10. The variations of $\frac{5}{2}n$ things taken 3 together : variations of $\frac{3}{2}n$ things taken 3 together :: 145 : 2. Find n

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269. THEOREM III.—*The number of combinations of* n *things taken p together is* $\dfrac{n(n-1)(n-2)(n-3)\dots\dots(n-p+1)}{1.2.3.4.\dots p}$

DEMONSTRATION. The number of combinations of n things two and two together is evidently only half as great as the number of variations of n things two together. Since each combination ab gives two variations, ab, ba, hence the combinations of n things two together is $\dfrac{n(n-1)}{2}$.

Again, since there are $n(n-1)(n-2)$ variations of n things taken three together, and each combination of three things admits of $1.2.3$ variations, it is evident that there are $1.2.3$ times as many variations of n things taken three together as of combinations taken three together, and consequently the number of combinations is $\dfrac{n(n-1)(n-2)}{1.2.3}$.

Similarly, the variations of n things taken p together is $n(n-1)(n-2)\ldots.(n-p+1)$, and every combination of p things will make $1.2.3\ldots.p$ variations. Hence there are $1.2.3\ldots.p$ times as many variations as combinations of n things taken p together, and consequently the number of combinations is $\dfrac{n(n-1)(n-2)\ldots.(n-p+1)}{1.2.3\ldots.p}$.

270. THEOREM IV.—*The number of combinations of* n *things taken* n − p *at a time is equal to the number of them taken* p *at a time.*

DEMONSTRATION. It has been shown by last theorem that the number of combinations of n things taken p together is $\dfrac{n(n-1)(n-2)\ldots.(n-p+1)}{1.2.3\ldots.p}$, and multiplying both numerator and denominator of this expression by $1.2.3\ldots.(n-p)$ we find that it $= \dfrac{n(n-1)(n-2)\ldots\ldots(n-p+1)\times(n-p)\ldots\ldots3.2.1}{1.2.3\ldots\ldots p\times 1.2.3\ldots\ldots(n-p)}$

$$= \frac{n(n-1)(n-2)\ldots\ldots.3.2.1}{\underline{|p}\ \underline{|n-p}} = \frac{\underline{|n}}{\underline{|p}\ \underline{|n-p}}$$

Now putting $n-p$ for p in this result, as may evidently be done, since the expression holds for all values of p which are less than n, we have $n-p = n-n+p = p$ and consequently

$$C_p = \frac{\underline{|n}}{\underline{|p}\ \underline{|n-p}} = \frac{\underline{|n}}{\underline{|n-p}\ \underline{|p}} = \ = C_{n-p}$$

that is, the C_p of n things $= C_{n-p}$ of the same n things.

Hence if $p > \frac{1}{2}n$, the number of combinations is more easily found by the supplemental formula, i. e., taken C_{n-p} instead of C_p.

NOTE.—The truth of this principle is also evident from the fact that if, from n things p be taken, $(n-p)$ things will always remain, and hence for every different set containing p things there will be a different set left containing $n-p$ things, and consequently the number of the former equals the number of the latter.

Cor. 1. Hence representing combinations of n things, 1, 2, 3, &c., p together, by C_1, C_2, C_3, &c., C_p we have

$$C_1 = \frac{n}{1},\ C_2 = \frac{n(n-1)}{1.2},\ C_3 = \frac{n(n-1)(n-2)}{1.2.3}, \text{ \&c.}$$

Cor. 2. To find the sum of all the combinations that can be made of n things taken 1, 2, 3, &c., n together, we proceed as follows :—

It will be shown hereafter that $\frac{n}{1}$, $\frac{n(n-1)}{1.2}$, $\frac{n(n-1)(n-2)}{1.2.3}$ &c., are the coefficients in the expansion of the binomial $(1+x)^n$, so that $(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + \text{\&c.} + C_nx^n$.

Now writing 1 for x we have

$$(1+1)^n = 2^n = 1 + C_1 + C_2 + C_3 + \text{\&c.} + C_n.$$

Hence $2^n - 1 = C_1 + C_2 + C_3 + \text{\&c.} + C_n$, or the sum of all the combinations which can be made of n things taken 1, 2, 3, &c., n together $= 2^n - 1$.

Ex. 1. Required the number of combinations of 22 things taken 5 together.

OPERATION.

Here $n = 22$ and $p = 5$

$$C_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} = \frac{22.21.20.19.18}{1.2.3.4.5}$$

$= 22.21.19.3 = 26334.$

Ex. 2. How many combinations can be made out of 23 things taken 19 together?

OPERATION.

Here $n = 23$ and $p = 19$, and consequently $n - p = 4$

$$C_p = C_{n-p} \text{ or } C_{19} = C_4 = \frac{23.22.21.20}{1.2.3.4} = 8855.$$

Ex. 3. What is the sum of all the combinations which can be made out of 10 things taken 1, 2, 3, &c., 10 at a time.

OPERATION.

$$C_1 + C_2 + C_3 + C_4 + \&c. + C_{10} = 2^{10} - 1 = 1024 - 1 = 1023.$$

Ex. 4. Out of 10 consonants and 3 vowels how many words each containing two vowels and four consonants can be found?

OPERATION.

10 consonants combined together 4 and 4 will give $\frac{10.9.8.7}{1.2.3.4}$ = 210 combinations; and similarly the combination of three vowels two together = $\frac{3.2}{1.2} = 3$. Hence the combinations of the 10 consonants and 3 vowels = $210 \times 3 = 630$.

But each of these combinations of 6 letters will furnish 1.2.3.4.5.6 = 720 permutations each, forming a different word. Hence the entire number of words formed will be $630 \times 720 = 453600$.

Ex. 5. How often may a *different* guard of 4 men be posted out of 50? On how many occasions would a given man be selected?

OPERATION.

$$C_4 = \frac{50.49.48.47}{1.2.3.4} = 230300$$

Taking away one man there remains 49, and the question now becomes, how many combinations may be formed of 49 men taken three together.

$C_3 = \frac{49.48.47}{1.2.3} = 18424$, to each of which the reserved man may be attached.

EXERCISE LXIII.

1. How many combinations may be made of 10 things taken 3 together? How many 5 together? How many 8 together?

2. How many combinations can be formed out of 15 things 5 together? How many 7 together? How many 12 together?

3. How many different classes of 5 children can be formed out of a school containing 12 children?

4. The whole number of combinations of $2n$ things is 513 times the whole number of combinations of n things; find n.

5. From a company of 36 policemen 5 are taken every night for special duty. On how many different nights may a different selection be made; and in how many of these will any particular man be engaged?

6. How many words of 7 letters can be made out of the 26 letters of the alphabet, with three out of the five vowels in every word?

7. In how many ways can 16 persons be seated at a round table so that all shall not have the same neighbours in any two arrangements?

8. If the permutations of n things 3 together : combinations of n things 4 together :: 6 : 1. Find n.

9. The number of permutations of n things p together is 10 times as great as their number taken $p - 1$ together, and the number of combinations p together : number $p - 1$ together :: 5 : 3. Find n and p.

10. In how many ways may n persons be arranged in a circle?

11. With ten flags representing the 10 numerals, how many signals can be formed, each representing a number, and not consisting of more than five flags?

12. How many different sums can be formed with a guinea, a half guinea, a crown, a half-crown, a shilling, a sixpence, a penny, a halfpenny, and a farthing?

SECTION XII.

BINOMIAL THEOREM.

271. The Binomial Theorem is a general formula invented by Sir Isaac Newton, for the purpose of expeditiously involving any binomial to any power. The formula is expressed as follows:

$$(a+x)^n = a^n + \frac{n}{1} a^{n-1} x + \frac{n(n-1)}{1.2} a^{n-2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} a^{n-3} x^3$$

$+ \&c.$, the $(r+1)$ *th* term being $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots.r} a^{n-r} x^r$,

Where $(a + x)$ is the given binomial, n, the exponent of the required power may be any quantity positive or negative, integral or fractional, and r any positive integer whatever.

NOTE 1.—The $(r + 1)$th term as above is commonly called the *general term* of the expansion.

NOTE 2.—The coefficients of x, x^2, x^3 &c., x^r in the above expansion are, when n is a positive integer, merely the general expressions for the number of combinations of n things taken 1, 2, 3, &c., r together (See Art. 269), and we shall therefore use the expressions C_1, C_2, C_3 &c., C_r to represent these coefficients, so that the formula given above may be written $(a + x)^n = a^n + C_1\ a^{n-1}\ x + C_2\ a^{n-2} x^2 + \text{\&c.}, + C_r\ a^{n-r}\ x^r + \text{\&c.}$

272. Since in the formula $(a + x)^n = a^n + C^1\ a^{n-1}\ x + C^2\ a^{n-2}x^2$ + &c., a and x represent any quantities whatever, we may write $-x$ in place of x and we thus obtain:—

$$(a - x)^n = a^n + C_1\ a^{n-1}\ (-x) + C_2\ a^{n-2}\ (-x)^2 + \text{\&c.}$$
$$= a^n - C_1\ a^{n-1}\ x + C_2\ a^{n-2}\ x^2 - \text{\&c.}$$

The terms being alternately *plus* and *minus*.

Cor. If $a = 1$, $(a \pm x)^n = (1 \pm x)^n = 1 \pm C_1\, x + C_2\, x^2 \pm C_3\, x^3 + C_4\, x^4 \pm \text{\&c.}$

273. THEOREM I.—*The Binomial Theorem is true in all cases when* n *is positive and integral.*

DEMONSTRATION.—By actual multiplication it appears that:—

$$(x + a)(x + b) = x^2 + (a + b)\ x + ab.$$
$$(x + a)(x + b)(x + c) = x^3 + (a + b + c)\ x^2 + (ab + ac + bc)\ x + abc.$$
$$(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)\ x^3 + (ab + ac + bc + ad + bd + cd)\ x^2 + (abc + acd + bcd + abd)\ x + abcd.$$

Now it is evident that in these results the following laws hold:—

I. *The number of terms in the right hand side, is one more than the number of binomial factors which are multiplied together.*

II. *The exponent of x in the* 1st *term = the number of binomial factors, and it decreases by unity in each succeeding term.*

III. *The coefs. of* 1st *terms = unity; coefs. of* 2nd *terms = sum of* 2nd *terms of all the binomial factors; coefs. of* 3rd *terms = the sum of all the products of the* 2nd *terms of the binomial factors taken two at a time; coefs. of* 4th *terms =*

sum of all the products of same second terms taken three at a time and so on; the last term is the product of all the second terms of the binomial factors taken all together.

Let us assume then that these laws of formation in the product hold for $n - 1$ binomial factors $(x + a)$, $(x + b)$, $(x + c)$, &c.

So that $(x + a)(x + b)(x + c)$ &c......$(x + k)$

$= x^{n-1} + A\,x^{n-2} + B\,x^{n-3} + Cx^{n-4} + \&c.\ldots + K.$

where $A = a + b + c + \ldots\ldots k$; $B = ab + ac + bc + \&c.$

$C = abc + acd + \&c.$

&c. = &c.

$K = abcd \ldots k.$

Then introducing a new factor $x + l$ we have:

$(x + a)(x + b)$ &c....... $(x + k)\ (x + l) = x^n + (A + l)\ x^{n-1} + (B + lA)\ x^{n-2} + \&c.\ldots\ldots + Kl.$

Wherefore $A + l = a + b + c \ldots\ldots + k + l$

$B + lA = ab + ac + bc + \ldots\ldots + al + bl + \ldots\ldots kl.$

&c. = &c.

$Kl = abcd \ldots\ldots kl.$

That is $A + l$ = sum of all the second terms of the binomial factors.

$B + lA$ = sum of all the products of the second terms a, b, c, ... l taken two at a time. And so on, and

Kl = product of the second terms when taken all together.

Hence if the laws indicated hold good when $n - 1$ factors are multiplied together, they hold good also when n factors are multiplied together. But we have shown that they hold good when 4 factors are multiplied together, therefore they hold when 5 factors are multiplied together, and therefore also for 6 and so on, and hence generally for any number whatever.

Now let $a = b = c = d = \&c.$

Then $A = a + a + a + \ldots\ldots$ to n terms $= na$.

$B = a^2 + a^2 + \ldots\ldots$ &c., to a number of terms = to the No. of combinations of n things taken two together $= \dfrac{n(n-1)}{1.2}\,a^2.$

$C = a^3 + a^3 +$ &c., to n number of terms = to the No. of combinations of n things taken three together
$$= \frac{n(n-1)(n-2)}{1.2.3} a^3. \qquad \text{And so on.}$$

$K = a.a.a.a$ to n factors $= a^n$.

Also, $(x+a)(x+b)(x+c)\ldots\ldots$ &c., becomes $(x+a)(x+a)$ $\ldots\ldots n$ terms $= (x+a)^n$.

$$\therefore\ (x+a)^n = x^n + \frac{n}{1} a x^{n-1} + \frac{n(n-1)}{1.2} a^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3} a^3 x^{n-3} + \cdots\cdots + a^n.$$

274. THEOREM II.—*The Binomial Theorem holds for all values of* n *either positive or negative, integral or fractional.*

DEMONSTRATION. (EULER'S.) It has been already shewn that when n and m are positive integers,

$$\text{(I.)}\ (1+x)^m = f(m) = 1 + \frac{m}{1} x + \frac{m(m-1)}{1.2} x^2 + \frac{m(m-1)(m-2)}{1.2.3} x^3 + \text{\&c.}$$

$$\text{(II.)}\ (1+x)^n = f(n) = 1 + \frac{n}{1} x + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \text{\&c.},$$

where $f(m)$ and $f(n)$ are symbols used to denote the series $1 + \frac{m}{1} x + \frac{m(m-1)}{1.2} x^2 +$ &c. and $1 + \frac{n}{1} x + \frac{n(n-1)}{1.2} x^2 +$ &c.

Hence *whatever* may be the values of m and n,

$$\left\{1 + \frac{m}{1} x + \frac{m(m-1)}{1.2} x^2 + \text{\&c.}\right\}\left\{1 + \frac{n}{1} x + \frac{n(n-1)}{1.2} x^2 + \text{\&c}\right\} = f(m) \times f(n).$$

But the product of these two series will evidently be a series of the form of $1 + ax + bx^2 + cx^3 +$ &c., ascending regularly by the integral powers of x, the letters a, b, c, &c., being used to represent the coefficients, found by addition, of x, x^2, x^3, &c.

Now it is evident* that the product of these two series must be of the *same form* whether m and n are positive or negative, integral or fractional. Whatever therefore be the *forms* assumed by a, b, c, &c., when m and n are positive integers, they will remain the same when m and n become fractional or negative.

But when m and n are positive and integral we have seen that by multiplying I and II together we get

$$f(m) \times f(n) = (1+x)^m \times (1+x)^n = (1+x)^{m+n} = 1 + ax + bx^2 + cx^3 + \&c.$$

$$= 1 + \frac{m+n}{1}x + \frac{(m+n)(m+n-1)}{1.2}x^2 + \frac{(m+n)(m+n-1)(m+n-2)}{1.2.3}x^3 + \&c.$$

$= f(m+n)$ by the notation adopted.

(III). $\therefore$ Generally $f(m) \times f(n) = f(m+n)$ for all values of m and n.

And since this is true for all values of m and n, for n we may write $n+r$, then $f(m+n+r) = f(n+r) \times f(m) = f(m) \times f(n) \times f(r)$.

Similarly $f(m+n+r+s+\ldots\ldots) = f(m) \times f(n) \times f(r) \times f(s) \times \ldots\ldots$ *i. e.* the product of *two or more* such series as that denoted by $f(m)$ produces another series of precisely the *same form*.

Now let $m = n = r = s = \&c., = \frac{p}{q}$ where p and q are positive integers, and suppose the number of terms to be q.

Then $f\left(\frac{p}{q} + \frac{p}{q} + \frac{p}{q} + \&c., \text{ to } q \text{ terms}\right) = f\left(\frac{p}{q}\right) \times f\left(\frac{p}{q}\right) \times f\left(\frac{p}{q}\right) \times \ldots\ldots$ to q factors.

---

* The product of two algebraic factors is not altered *in form* by any variation in the value or nature of the factors. Thus $(x+a)(x+b) = x^2 + (a+b)x + ab$ for all values of x, a and b. So in the above, although by changing the values of m and n we alter the values of b, c, &c., yet their forms, *i.e.* the manner in which m and n enter the series, remain the same.

$\therefore \int(p) = \left\{\int\left(\frac{p}{q}\right)\right\}^q$. But since p is a positive integer,
$\int(p) = (1+x)^p \therefore (1+x)^p = \left\{\int\left(\frac{p}{q}\right)\right\}^q \therefore (1+x)^{\frac{p}{q}} = \int\left(\frac{p}{q}\right)$

or $(1+x)^{\frac{p}{q}} = 1 + \frac{\frac{p}{q}}{1}x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1.2}x^2 + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{1.2.3}x^3 + \&c.$

by the notation adopted.

Thus the Theorem is proved for a fractional index.

Again in (III) put $m = -n$.

Then $\int(n) \times \int(-n) = \int(n-n) = \int(0) = 1 \therefore$ the assumed series becomes 1 when $m = 0$.

And since $\int(n) \times \int(-n) = 1$ dividing each by $\int(n)$

$$\int(-n) = \frac{1}{\int(n)} = \frac{1}{(1+x)^n} \text{ since } n \text{ is positive.}$$

And $\frac{1}{(1+x)^n} = (1+x)^{-n}$ by Art. 165 $\therefore (1+x)^{-n}$

$= \int(-n) = 1 + \left(\frac{-n}{1}\right)x + \frac{(-n)(-n-1)}{1.2}x^2 + \frac{-n(-n-1)(-n-2)}{1.2.3}$
$x^2 + \&c.$

Thus the theorem is also proved when n is any negative quantity.

275. From this theorem then it appears that :—

I. $(1 \pm x)^n = 1 \pm \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 \pm \frac{n(n-1)(n-2)}{1.2.3}x^3 + \&c.$

II. $(1 \pm x)^{-n} = 1 \pm \frac{-n}{1}x + \frac{-n(-n-1)}{1.2}x^2 \pm \frac{-n(-n-1)(-n-2)}{1.2.3}x^3$
$+ \&c.$

$$= 1 \mp \frac{n}{1}x + \frac{n(n+1)}{1.2}x^2 \mp \frac{n(n+1)(n+2)}{1.2.3}x^3 + \&c.$$

III. $(1 \pm x)^{\frac{p}{q}} = 1 \pm \frac{\frac{p}{q}}{1}x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1.2}x^2 \pm \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{1.2.3}x^3$
$+ \&c.$

$$= 1 \pm \frac{p}{q} x + \frac{\frac{p}{q} \cdot \frac{p-q}{q}}{1.2} x^2 - \frac{\frac{p}{q} \cdot \frac{p-q}{q} \cdot \frac{p-2q}{q}}{1.2.3} x^3 + \&c.$$

$$= 1 \pm \frac{p}{q} x + \frac{p(p-q)}{1.2.q^2} {}^2 \pm \frac{p(p-q)(p-2q)}{1.2.3.q^3} x^3 + \&c.$$

IV. $(1 \pm x)^{-\frac{p}{q}} = 1 \pm \frac{-\frac{p}{q}}{1} \frac{-\frac{p}{q}\left(-\frac{p}{q}-1\right)}{1.2} x^2 \pm \frac{-\frac{p}{q}\left(-\frac{p}{q}-1\right)\left(-\frac{p}{q}-2\right)}{1.2.3} x^3$ $+ \&c.$

$$= 1 \mp \frac{p}{q} x + \frac{p(p+q)}{1.2.q^2} x^2 \mp \frac{p(p+q)(p+2q)}{1.2.3.q^3} x^3 + \&c.$$

And these reduced general expressions should be carefully noticed by the student, and used as formulæ for the expansion of binomials according as n is positive or negative, integral or fractional.

NOTE.—No examples with n integral and positive are given, as there are a number such in Exercise XXXVIII.

Ex. 1. $(1+x)^{10} = 1 + \frac{10}{1} x + \frac{10.9}{1.2} x^2 + \frac{10.9.8}{1.2.3} x^2 + \frac{10.9.8.7}{1.2.3.4}$ $x^4 + \&c.$

$$= 1 + 10x + 45x^2 + 120x^3 + 210x^4 + \&c.$$

Ex. 2. $(1+x)^{-5} = 1 - \frac{5}{1} x + \frac{5.6}{1.2} x^2 - \frac{5.6.7}{1.2.3} x^3 + \frac{5.6.7.8}{1.2.3.4}$ $x^4 + \&c.$

$$= 1 - 5x + 15x^2 - 35x^3 + 70x^4 - \&c.$$

Ex. 3. $(1-x)^{-1} = 1 + \frac{1}{1} x + \frac{1.2}{1.2} x^2 + \frac{1.2.3}{1.2.3} x^3 + \frac{1.2.3.4}{1.2.3.4} x^4$ $+ \&c.$

$$= 1 + x + x^2 + x^3 + x^4 + \&c.$$

NOTE.—Hence it appears that in all cases when n is integral if the sign of the exponent and that connecting the terms of the binomial are both like, *i. e.* either both *plus* or both *minus*, the signs of the expansion are all *plus*, but if unlike, the signs of the expansion are *plus* and *minus* alternately.

Ex. 4. $(1+x)^{\frac{3}{5}} = 1 + \frac{3}{5} x + \frac{3(3-5)}{1.2.25} x^2 + \frac{3(3-5)(3-10)}{1.2.3.125} x^3 +$ $\frac{3(3-5)(3-10)(3-15)}{1.2.3.4.625} x^4 + \&c.$

$$= 1 + \frac{3}{5} x + \frac{3 \times -2}{1.2.25} x^2 + \frac{3 \times -2 \times -7}{1.2.3.125} x^3 + \frac{3 \times -2 \times -7 \times -12}{1.2.3.4.625} x^4 + \&c.$$

$$= 1 + \frac{3}{5} - \frac{3}{25}x^2 + \frac{7}{125}x^3 - \frac{21}{625}x^4 + \&c.$$

Ex. 5. $(1-x)^{-\frac{3}{2}} = 1 + \frac{3}{2}x + \frac{3(3+2)}{1.2.4}x^2 + \frac{3(3+2)(3+4)}{1.2.3.8}x^3$
$+ \frac{3(3+2)(3+4)(3+6)}{1.2.3.4.16}x^4 + \&c.$

$$= 1 + \frac{3}{2}x + \frac{3.5}{1.2.4}x^2 + \frac{3.5.7}{1.2.3.8}x^3 + \frac{3.5.7.9}{1.2.3.4.16}x^4 + \&c.$$

$$= 1 + \frac{3}{2}x + \frac{15}{8}x^2 + \frac{35}{16}x^3 + \frac{315}{128}x^4 + \&c.$$

Ex. 6. $(a+2x)^{-2} = \left\{a\left(1+\frac{2x}{a}\right)\right\}^{-2} = a^{-2}(1+2a^{-1}x)^{-2}$

$= a^{-2}\left\{1 - \frac{2}{1}(2a^{-1}x) + \frac{2.3}{1.2}(2a^{-1}x)^2 - \frac{2.3.4}{1.2.3}(2a^{-1}x)^3 + \frac{2.3.4.5}{1.2.3.4}\right.$
$(2a^{-1}x)^4$
$= a^{-2}\{1 - 4a^{-1}x + 12a^{-2}x^2 - 32a^{-3}x^3 + 80a^{-4}x^4 - \&c.\}$
$= a^{-2} - 4a^{-3}x + 12a^{-4}x^2 - 32a^{-5}x^3 + 80a^{-6}x^4 - \&c.$

Ex. 7. $(a^2+x^2)^{-\frac{3}{4}} = \{a^2(1+a^{-2}x^2)\}^{-\frac{3}{4}} = a^{-\frac{3}{2}}(1+a^{-2}x^2)^{-\frac{3}{4}}$

$= a^{-\frac{3}{2}}\left\{1 - \frac{3}{4}(a^{-2}x^2) + \frac{3.7}{1.2.16}(a^{-2}x^2)^2 - \frac{3.7.11}{1.2.3.64}(a^{-2}x^2)^3\right.$
$+ \frac{3.7.11.15}{1.2.3.4.256}(a^{-2}x^2)^4$

$= a^{-\frac{3}{2}}\{1 - \frac{3}{4}a^{-2}x^2 + \frac{21}{32}a^{-4}x^4 - \frac{77}{128}a^{-6}x^6 + \frac{1155}{2048}a^{-8}x^8 + \&c.\}$
$= a^{-\frac{3}{2}} - \frac{3}{4}a^{-\frac{7}{2}}x^2 + \frac{21}{32}a^{-\frac{11}{2}}x^4 - \frac{77}{128}a^{-\frac{15}{2}}x^6 + \frac{1155}{2048}a^{-\frac{19}{2}}x^8$
$- \&c.$

EXERCISE LXIV.

Expand to five terms each of the following expressions :—

1. $(1+x)^{-3}$
2. $(1+x)^{-2}$
3. $(1-2x)^{-1}$
4. $(1-\frac{1}{2}x)^{-5}$
5. $(1+3x)^{-2}$
6. $\frac{1}{(1-2x)^{\frac{1}{5}}}$
7. $\frac{1}{(1-x)^4}$
8. $(1-4x)^{\frac{1}{2}}$
9. $(1+x)^{-\frac{2}{3}}$
10. $(1-\frac{3}{4}x)^{\frac{4}{5}}$
11. $(1+\frac{2}{3}x)^{\frac{1}{3}}$
12. $\frac{1}{(1-x)^{\frac{4}{5}}}$
13. $(a-x^2)^{-3}$
14. $(a^2+x^3)^{-1}$
15. $(a^{\frac{1}{2}}-x^{\frac{1}{3}})^{-2}$
16. $(a^4-x^3)^{\frac{2}{3}}$
17. $(a^3+x^{-2})^{-4}$
18. $(a^{\frac{1}{5}}-x^{-\frac{1}{5}})^{-\frac{1}{3}}$
19. $(a^2m-x^{\frac{1}{2}})^{-\frac{2}{3}}$
20. $\frac{1}{(a+x^{-3})^{-\frac{2}{5}}}$
21. $\frac{1}{\sqrt{a-bx}}$

276. THEOREM III.—*In the expansion of* $(1 + x)^n$ *there are only* $n + 1$ *terms, when the exponent is positive and integral.*

DEMONSTRATION.—The coefficient of the $(r + 1)$th term is $C_r = \frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1)}{|\underline{r}}$. Now if r be such that $n - r + 1 = 0$, then the $(r + 1)$th and all following terms vanish, and the series will terminate with the rth term. But if $n - r + 1 = 0$, $r = n + 1$ and the $(n+1)$th term is the last term of the series.

NOTE.—If n is negative or fractional, the series never ends, but may be continued to an infinite number of terms, since as r is necessarily integral and positive, we can then find no value for r which will render $n - r + 1 = 0$.

277. THEOREM IV.—*In the expansion of* $(1 + x)^n$ *when* n *is positive and integral, the coefficients of terms equally distant from beginning and end are the same.*

DEMONSTRATION.—The $(r + 1)$th term from the end having r terms after it is the same as the $\{(n + 1) - r\}$ th term from the beginning, i. e., is the same as the $(n - r + 1)$th term from the beginning. And since, Art. 271, the coef. of the $(r + 1)$th term is C_r writing $n - r$ for r the coef. of the $(n - r + 1)$th term will be C_{n-r}.

But it has already been shown (Art. 270) that

$$C_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots.\ r} = \frac{|\underline{n}}{|\underline{r}\ |\underline{n-r}} = \frac{|\underline{n}}{|\underline{n-r}\ |\underline{r}} = C_{n-r},$$

that is the coef. of the $(r + 1)$ from the beginning = coef. of $(r + 1)$ term from the end.

278. *To find the general term of the expansion of* $(a + x)^n$.

In writing down any term of the expansion of $(1 + x)^n$, say the 5th term, so as to exhibit the factors of the coefficient thus,

$\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}x^4$, we observe

1. The numerator added to denominator of each factor $= n + 1$.

II. The number of such factors is one less than the number of the term.

III. The exponent of x is equal to the denom. of last factor.

IV. The exponent of $a = n -$ (the exponent of x).

Hence the $(r+1)$th or the general term of the expansion $=$
$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r} a^{n-r}x^r.$$

279. The student must note the following points with respect to this general term :—

I. The gen. term of $(1+x)^n$ when n is a positive integer, is as above.

II. When n is positive, the gen. term of $(1-x)^n = C_r(-x)^r$
$$= C_r(-1)^r x^r = (-1)^r C_r x^r = (-1)^r \left(\frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r} x^r\right)$$
where $(-1)^r$ will of course be positive or negative according as r is even or odd, that is, according as $r+1$, the number of the term, is odd or even.

III. If n be negative, the general term of $(1+x)^{-n} =$
$$\frac{-n(-n-1)\ldots-\{n-(r+1\}}{\lfloor r} = (-1)^r \left(\frac{n(n+1)\ldots(n+r-1)}{\lfloor r} x^r\right)$$

IV. If n is negative, the general term of $(1-x)^{-n} = (-1)^r \times$
$$C_r(-x)^r = (-1)^r(-1)^r\left(\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}\right)x^r$$
$$= \frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r} x^r.$$ Since $(-1)^r \times (-1)^r = (-1)^{2r} = +1$.

When the exponent is fractional, the sign of the general term is subject to the same laws, and C_r may be written as in III and IV on pages 226, 227. Thus the general term of

V. $(1+x)^{\frac{p}{q}} = \dfrac{p(p-q)(p-2q)\ldots\{p-(r-1)q\}}{\lfloor r \times q^r} x^r$

VI. $(1-x)^{-\frac{p}{q}} = (-1)^r\left(\dfrac{p(p+q)(p+2q)\ldots\{p+(r-1)q\}}{\lfloor r \times q^r} x^r\right)$

VII. $(1+x)^{-\frac{p}{q}} = \dfrac{p(p+q)(p+2q)\ldots\{p+(r-1)q\}}{\lfloor r \times q^r} x^r$

VIII. $(1-x)^{\frac{p}{q}} = (-1)^r \left(\frac{p(p-q)(p-2q)\ldots\{p-(r-1)q\}}{\lfloor r \times q^r} x^r\right)$

Ex. 1. Find the general terms in the expansions of $(1+x)^8$, $(1-x^2)^{-\frac{1}{2}}$, $(a^2-x^2)^{-\frac{3}{4}}$, $(1+3x)^{-2}$.

$$\text{G. T. of } (1+x)^8 = +\frac{8.7.6\ldots(8-r+1)}{1.2.3\ldots r}x^r = +\frac{8.7\ldots(9-r)}{\lfloor r}x^r$$

$$\text{G. T. of } (1-x^2)^{-\frac{1}{2}} = +\frac{1.3.5\ldots\{1+(r-1)2\}}{\lfloor r \times 2^r}(x^2)^r$$
$$= +\frac{1.3.5\ldots(2r-1)}{\lfloor r \times 2^r}x^{2r}$$

$$\text{G. T. of } (a^2-x^2)^{-\frac{3}{4}} = a^{-\frac{3}{2}}(1-a^{-2}x^2)^{-\frac{3}{4}} =$$
$$+ a^{-\frac{3}{2}}\left\{\frac{3.7.11\ldots\{3+(r-1)4\}}{\lfloor r \times 4^r}(a^{-2}x^2)^r\right. =$$
$$+ a^{\frac{3}{2}}\ \frac{3.7.11\ldots\ldots(4r-1)}{\lfloor r \times 4^r}a^{2r}x^{2r}.$$

$$\text{G. T. of } (1+3x)^{-2} = (-1)^r\left(\frac{2.3.4\ldots(2+r-1)}{\lfloor r}\right)(3x)^r = (-1)^r$$
$$\left(\frac{2.3.4\ldots\ldots(r+1)}{\lfloor r}\right)3^r x^r = (-1)^r(r+1)3^r x^r.$$ Since $\lfloor r$ in den. cancels $1.2.3\ldots r = \lfloor r$ in the numerator.

Ex. 2. Find the general term in the expansion of $(1+x)^{\frac{5}{3}}$

$$\text{G. T. of } (1+x)^{\frac{5}{3}} = \frac{5.2.-1.-4\ldots\{5-(r-1)3\}}{\lfloor r \times 3^r}x^r$$
$$= (-1)^r\frac{\{5.2.1.4\ldots(3r-8)\}}{\lfloor r \times 3^r}x^r$$

NOTE.—In the above expression for the general term it will be observed that we change all the negative signs in the numerator, and then prefix a power of (-1). Now if all the factors in the numerator are negative, $(-1)^r$ is the prefix, and if any even numbers of negative factors are changed to positive, $(-1)^r$ is still the prefix, but if any odd number of them is changed, the sign of the product of the whole, i. e. of the general term. is altered, and becomes $(-1)^{r+1}$. In the expansion of $(1+x)^{\frac{p}{q}}$ therefore the sign of the general term is $(-1)^r$ or $(-1)^{r+1}$, according as the number of positive factors is even or odd.

In the expansion of $(1-x)^{\frac{p}{q}}$ the general term will of itself involve $(-1)^r$, and this taken in connection with the above renders the sign of the general term $(-1)^{2r} = 1$ or $(-1)^{2r+1} = -1$ according as the number of positive factors is even or odd.

REMARK.—In the above paragraph the general term merely expresses any term after negative factors begin to appear in the numerator.

Ex. 3. Find the general term of $(1-x)^{\frac{3}{5}}$

$$\text{G. T. of } (1-x)^{\frac{3}{5}} = \frac{3 \cdot -2 \cdot -7 \ldots \{3-(r-1)5\}}{\lfloor r \times 5^r} x^r$$

$$= \frac{3 \cdot 2 \cdot 7 \ldots\ldots (5r-8)}{\lfloor r \times 5^r} x^r$$

Ex. 4. Find the 8th term of the expansion of $(1+x)^{-4}$

Since the general term $= (r+1)$th term $=$ 8th term, $r = 7$

$$\text{Formula II. 8th term} = (-1)^7 \left(\frac{4.5.6.7.8.9.10.11}{1.2.3.4.5.6.7}\right) x^8$$
$$= -1320x^8$$

Ex. 5. Find the 5th term of the expansion of $(1-x)^{-\frac{1}{2}}$

$$\text{Formula VII. 5th term} = \frac{1.3.5\ldots\{1+(4-1)2\}}{\lfloor 4 \times 2^4} x^4$$

$$= \frac{1.3.5.7}{1.2.3.4 \times 16} x^4 = \frac{35}{128} x^4$$

Ex. 6. Find the 7th term of the expansion of $(1-\frac{1}{3}x)^{11}$

$$\text{Formula II. 7th term} = (-1)^6 \frac{11.10.9.8.7.6}{1.2.3.4.5.6} (\tfrac{1}{3}x)^6$$

$$= +462 \times \frac{x^6}{729} = \frac{462}{729} x^6 = \frac{154}{243} x^6$$

Ex. 7. Find the 6th term of the expansion of $(1-x)^{\frac{7}{5}}$

$$\text{Formula VIII. } \frac{7.2.-3\ldots\{7-(5-1)5\}}{\lfloor 5 \times 5^5} x^5 = \frac{7.2.3\ldots\{(4\times5)-7\}}{\lfloor 5 \times 5^5} x^5$$

$$= +\frac{7.2.3.8.13}{1.2.3.4.5 \times 6125} x^5 = +\frac{182}{30625} x^5$$

Since there are two positive factors in the first expression, The sign is $(-1)^{2r} = +1$, see note above.

Ex. 8. Find the 11th term of the expansion of $(a^{-\frac{1}{2}} + x^2)^{\frac{11}{4}}$

$$(a^{-\frac{1}{2}} + x^2)^{\frac{11}{4}} = \left\{a^{-\frac{1}{2}}\,(1 + a^{\frac{1}{2}}x^2)\right\}^{\frac{11}{4}} = a^{-\frac{11}{8}}\left(1 + a^{\frac{1}{2}}\,x^2\right)^{\frac{11}{4}},$$

Then by formula v the 11th term.

$$= a^{-\frac{11}{8}}\left\{\frac{11\,.\,7\,.\,3\,.\,-1.-5\ldots.\{11-(10-1)4\}}{\underline{|10} \times 4^{10}}\right\}(a^{\frac{1}{2}}\,x^2)^{10}$$

$$= a^{-\frac{11}{8}} \times (-1)^{11}\left(\frac{11.7.3.1.5\ldots.(36-11)}{\underline{|10} \times 4^{10}}\right) a^5\,x^{20}$$

$$= a^{-\frac{11}{8}} \times -\frac{11.7.3.1.5.9.13.17.21.25}{1.2.3.4.5.6.7.8.9.10.1048576}\,a^5\,x^{20}$$

$$= a^{-\frac{11}{8}} \times -\frac{85085}{268435456}\,a^5x^{20} = -\frac{85085}{268435456}\,a^{\frac{29}{8}}\,x^{20}$$

280. *To find the sum of all the coefficients of* $(1 + x)^n$.

The Theorem $(1 + x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \&c.$, is true for all values of x. Let $x = 1$.

Then $(1 + 1)^n = 2^n = 1 + \frac{n}{1} + \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)}{1.2.3} + \&c.$

$\therefore$ 2^n = sum of all the coefficient of $(1 + x)^n$.

281. THEOREM V.—*The sum of the coefficients of the odd terms in the expansion of* $(1 + x)^n$ *is equal to the sum of the coefficients of the even terms.*

DEMONSTRATION.—Put $x = -1$ in the expansion of $(1 + x)^n$.

Then $(1 - 1)^n = 0^n = 0 = 1 - n + \frac{n(n-1)}{1.2} - \frac{n(n-1)(n-2)}{1.2.3} + \&c.$

Sum of coefficients of odd terms – sum of coef. of even terms = 0.

$\therefore$ Sum of coefficient of odd terms = sum of coefficients of even terms.

COR.—Since the sums are equal, each sum is evidently half of 2^n, Art. 280, and is therefore $= \frac{2^n}{2} = 2^{n-1}$

282. *To find the greatest term in the expansion of* $(a + x)^n$.

The $(r + 1)$th term $= \frac{n(n-1)(n-2)\ldots.(n-r+1)}{\underline{|r}}\,a^{n-r}\,x^r$

The rth term $= \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{\lfloor r-1} a^{n-r+1}x^{r-1}$.

Hence the $(r+1)$th term is obtained from the rth by multiplying the latter by $\dfrac{n-r+1}{r} \cdot \dfrac{x}{a}$. Consequently the rth term will be the greatest as soon as $\dfrac{n-r+1}{r} \cdot \dfrac{x}{a}$ becomes < 1.

That is as soon as $(n-r+1)\,x < ar$ or $r(a+x) > (n+1)\,x$.

That is as soon as $r > (n+1)\dfrac{x}{a+x}$.

r therefore must be the first whole number $> (n+1)\dfrac{x}{a+x}$.

If $(n+1)\dfrac{x}{a+x}$ is a whole number, then two terms are equal, and each is greater than any other term.

If n is negative, r is the first whole number equal to or next greater than $(n-1)\dfrac{x}{a-x}$.

Ex. 9. What is the sum of all the coefficients of $(1+x)^9$.
Here Art. 280, $2^n = 2^9 = 512$.

Ex. 10. What is the sum of all the odd coef. of $(1+x)^{15}$.
Here Art. 281, $2^{n-1} = 2^{15-1} = 2^{14} = 16384$.

Ex. 11. Which is the greatest term in the expansion of $(1+x)^{13}$ when $x = \cdot 3$.

Here r is the whole number equal to or first greater than $(13+1)\dfrac{\cdot 3}{1\cdot 3}$ or $14 \times \dfrac{3}{13}$ or $\dfrac{42}{13}$ which is 4, therefore the 4th term is the greatest.

Exercise LXV.

Find the general term and the 6th term of:—

1. $(1-x)^{-3}$ 2. $(1+x)^{-4}$ 3. $(1-x)^{-\frac{2}{3}}$ 4. $(1-x)^{\frac{4}{3}}$

5. $(1+x)^{-\frac{7}{2}}$ 6. $(1+x)^{-\frac{8}{3}}$ 7. $(a-x)^{-1}$ 8. $(a+\frac{1}{2}x)^{\frac{6}{5}}$

Find the general term and the 5th term of:—

9. $(1-2x)^{-2}$ 10. $(1+\frac{2}{3}x^2)^{-\frac{5}{2}}$

11. $(a^{-2}+x^{-\frac{2}{3}})^{-\frac{2}{5}}$ 12. $(a^{-\frac{1}{2}}-x^{-\frac{1}{2}})^{-2}$

Find the sum of all the coefficients of :—

13. $(1+x)^{10}$ 14. $(1+x)^7$ 15. $(1-x)^{13}$ 16. $(1+x)^{12}$.

17. Find the greatest term in the expansion of $(1+x)^4$ when $x = 2$.

18. Find the greatest term in the expansion of $(1+x)^{-5}$ when $x = \frac{1}{2}$.

19. Find the greatest term in the expansion of $(2a+x)^{20}$ when $a = \frac{1}{3}x = 1$.

20. Find the greatest term in the expansion of $(1+x)^{-7}$ when $x = \frac{3}{5}$.

SECTION XIII.

NOTATION AND PROPERTIES OF NUMBERS.

283. *Any number* N *may be expressed in the form of* $d_n r^n + d_{n-1} r^{n-1} + \&c. + d_3 r^3 + d_2 r^2 + d_1 r^1 + d^0$ *where* r *is a positive integer, and the coefficients* d^0, d_1, &c., d_{n-1}, d_n, *are also integers all less than* r, *the radix of the scale.*

For let N be divided by the *greatest power of r* it contains, and let the quotient be d_n, less of course than r, and let the remainder be N_1. Then $N = d_n r^n + N_1$.

Similarly let N_1 be divided by the *greatest power of r* it contains, and let the quotient be d^{n-1} with remainder N_2. Then $N_1 = d_{n-1} r^{n-1} + N_2$.

Similarly $N_2 = d_{n-2} r^{n-2} + N_3$, and so on, and continuing the process until the remainder becomes $< r$ = say d_0 we have

$$N = d_n r^n + d_{n-1} r^{n-1} + \ldots . \&c. + d_2 r^2 + d_1 r^1 + d_0.$$

Where any of the coefficients d_n, d_{n-1}, &c., d_3, d_2, d_1, d_0, may vanish, i. e., become $= 0$, but none can be $>$ or $= r$. In other words, these coefficients, or digits as they are called, may have any value from 0 to $r-1$ inclusive, and consequently in any scale r there occur r digits, including *zero*. (See National Arithmetic.)

284. *To express any number in any proposed scale :—*

Let N be the number and let r the radix of the proposed scale. Then by last Art., the given number may be written as =

$$d_n r^n + d_{n-1} r^{n-1} + \&c. + d_2 r^2 + d_1 r^1 + d_0.$$

Dividing this by r we get a complete quotient with remainder d^0, the right digit of the number in the proposed scale.

Dividing this complete quotient by r, we get another complete quotient with rem. d_1, which is the second digit of the number.

And proceeding thus as long as we get a quotient divisible by r, we obtain as remainders the successive digits of the number. (See Arithmetic.)

285. *To prove the rule for reducing a pure repetend to its equivalent vulgar fraction.*

Let R = the given repetend, and let it contain r digits, and let V = its value.

Then $V = \cdot RR$ &c. (I). Multiplying each by 10^r we have $10^r V = R\cdot RR$ &c. (II). Subtracting (I) from (II)

$$10^r V - V = R \therefore V(10^r - 1) = R \therefore V = \frac{R}{10^r - 1}.$$

But since r = the number of digits in the repetend, $10^r - 1$ will be as many 9's as there are digits in the repetend.

$$\therefore V = \frac{\textit{Repetend}}{\text{As many 9's as there are digits in } \textit{repetend}}.$$

286. *To prove the rule for reducing a mixed repetend to its equivalent vulgar fraction.*

Let V = the value of a mixed repetend in which F represents the finite part and R the repetend, and let F and R contain respectively f and r digits.

Then $V = \cdot FRR$ &c. Multiplying these by 10^{f+r} we have $10^{f+r} V = FR\cdot RR$ &c. (I). Also multiplying them by 10^f, $10^f V = F\cdot RR$ &c. (II). Subtracting II from I,

$$(10^{f+r} - 10^f) V = FR - F. \text{ That is, } V = \frac{FR - F}{10^f(10^r - 1)}$$

But 10^f is unity followed by as many ciphers as there are units in f, i. e., as many ciphers as there are digits in F, the

finite part, and $10^r - 1$ is as many 9's as there are units in r, i.e., as many 9's as there are digits in R, the repetend.

$$\therefore V = \frac{\text{Whole repetend minus the finite part.}}{\text{As many 9's as figures in repetend followed by as many 0's as figures in finite part.}}$$

287. THEOREM I.—*If from any number the sum of its digits be subtracted, the remainder is divisible by the radix of the scale decreased by unity.*

DEMONSTRATION.—Let r be the radix of the scale, and

let $a + br + cr^2 + dr^3 +$ &c. be the number.

Subtract $a + b + c + d +$ &c. the sum of the digits.

Then the rem. $= br - b + cr^2 - c + dr^3 - d +$ &c. $= b(r-1) + c(r^2-1) + d(r^3-1) +$ &c., which (Art. 80) is evidently divisible by $r - 1$ i.e., by the radix decreased by unity.

288. THEOREM II.—*If the sum of the digits of any number is divisible* (r − 1), *that is by the radix decreased by unity, then the number itself is divisible by one less than the radix.*

DEMONSTRATION.—For let N = the number and S = the sum of its digits, and since S is by hypothesis divisible by $(r-1)$ let $S = m(r-1)$. Then Theorem I, $N - S$ is also divisible by $r - 1$, $\therefore$ let $N - S = p(r-1)$.

Then by substitution we have $N - m(r-1) = p(r-1)$

$\therefore N = p(r-1) + m(r-1) = (r-1)(p+m)$, and since the right-hand member is evidently divisible by $r - 1$ $\therefore$ also the left-hand member N is divisible by $r - 1$.

Cor. In any scale such that $r - 1$ is divisible by 3, if the sum of the digits of any number be divisible by 3, the number itself is divisible by 3. For let N and S represent the number and the sum of its digits, and let $S = 3m$ and $r - 1 = 3q$.

Then $N - S = p(r-1) = 3pq \therefore N - 3m = 3pq \therefore N = 3(pq + m)$,
That is, N is divisible by 3.

Hence in the common scale a number is divisible by 3 or by 9, according as the sum of its digits is divisible by 3 or by 9.

289. THEOREM III.—*If from any number the sum of the digits standing in the odd places be subtracted, and to it the sum of the*

digits standing in the even places be added, then the result is divisible by the radix increased by unity.

DEMONSTRATION.—Let r be the radix and let the number be

$$a + br + cr^2 + dr^3 + er^4 + \&c.$$

$$\text{Add} - \underline{a + b - c + d - e + \&c.}$$

The result is $br + b + cr^2 - c + dr^3 + d + er^4 - e + \&c.$, which is equal to $b(r + 1) + c(r^2 - 1) + d(r^3 + 1) + e(r^4 - 1) + \&c.$

But $r + 1$, $r^2 - 1$, $r^3 + 1$, $r^4 - 1$, &c., are all (Art. 80) divisible by $r + 1$, $\therefore b(r + 1) + c(r^2 + 1) + d(r^3 + 1) + \&c.$ is divisible by $r + 1$.

Cor. Hence in the common scale any number answering the conditions given above is divisible by 11.

290. THEOREM IV.—*If in any number the sum of the digits standing in the even places be equal to the sum of the digits standing in the odd places, then the number is divisible by the radix increased by unity.*

Let N = the number, S = the sum of digits in the even places, and S_1 the sum of the digits in the odd places.

Then Theorem III, $N + S - S_1$ is divisible by $r + 1$. But since by hypothesis $S = S_1$, it follows that $S - S_1 = 0 \therefore N$ is divisible by $r + 1$.

291. *To prove the common rule for testing the accuracy of multiplication by casting out the 9's.*

DEMONSTRATION.—It follows from Theorem II. that any number in the common scale will leave the same remainder when divided by 9 that the sum of its digits will leave when divided by 9. Let then $9a + c$ be the multiplicand and $9b + d$ be the multiplier. Then $81ab + 9bc + 9ad + cd$ will be the product. Now if the sum of the digits inthe multiplicand be divided by 9, the rem. is c, if the sum of the digits in the multiplier is divided by 9 the rem. is d, and if the sum of the digits in the product be divided by 9, the rem. is evidently the same as the rem. obtained by dividing cd by 9.

292. THEOREM V.—*The product of any three consecutive numbers in the scale of* 10 *is divisible by* 1.2.3, i.e., *by* 6.

DEMONSTRATION.—Every number must be of the form of $3m$ or $3m + 1$, or $3m + 2$, because every number when divided by 3 must leave 0 or 1 or 2 as remainder.

$\therefore$ The product of any three consecutive numbers may be represented by $3m(3m + 1)(3m + 2)$. But $3m$ is a multiple of 3 and of the other factors $3m + 1$ or $3m + 2$ one must be even, and must therefore be divisible by 2, $\therefore 3m(3m + 1)(3m + 2)$ must be divisible by $1.2.3$, i.e., by 6.

293. THEOREM VI.—*The product of any* r *consecutive numbers is divisible by* $1.2.3 \ldots . r$.

DEMONSTRATION.—Let n be the least of the numbers, and let $\dfrac{n(n+1)(n+2)\ldots .(n+r-1)}{1.2.3.4\ldots .r}$ be represented by ${}^{n}P_r$ for all values of n and r.

$$\text{Then } {}_nP_r = \frac{n(n+1)\ldots .(n+r-2)}{1.2.3\ldots .(r-1)} \cdot \frac{n+r-1}{r} = {}_nP_{r-1}\left(\frac{n-1}{r} + 1\right)$$

$$= {}_nP_{r-1} \times \frac{n-1}{r} + {}_nP_{r-1} = \frac{(n-1)n(n+1)(n+2)\ldots .(n+r-2)}{1.2.3\ldots .r} + P_{r-1} = {}_{n-1}P_r + {}_nP_{r-1}.$$

Now if we assume that ${}_nP_{r-1}$ is an integer, or in other words that the product of any $(r-1)$ consecutive integers is divisible by $1.2.3\ldots .r$.

Then since as above shown ${}_nP_r = {}_{n-1}P_r + {}_nP_{r-1}$ we have ${}_nP_r = {}_{n-1}P_r + \text{int.}$, an integer for all values of n and r, and writing in succession $n-1$, $n-2\ldots .3.2$ for n we obtain

${}_{n-1}P_r = {}_{n-2}P_r + \text{int.}$,

${}_{n-2}P_r = {}_{n-3}P_r + \text{int.}$

&c. = &c.

${}_3P_r = {}_2P_r + \text{int.}$

${}_2P_r = {}_1P_r + \text{int.}$ Adding these equals and cancelling, we have ${}_nP_r = {}_1P_r +$ sum of integers, but ${}_1P_r = \dfrac{1.2.3.4\ldots .r}{1.2.3.4\ldots .r} = 1$.

$\therefore {}_nP_r = 1 +$ sum of integers = an integer.

Hence if ${}_nP_{r-1}$ is an integer, then also ${}_nP_r$ is an integer. But it has been shown Theorem V that ${}_nP_3$ is an integer therefore also ${}_nP_4$ is an integer, and therefore also ${}_nP_5$ and so on, $\therefore {}_nP_r$ is an integer, that is $n(n+1)(n+2)\ldots .(n+r-1)$ is divisible by $1.2.3\ldots .r$.

SECTION XIV.

INEQUALITIES, VANISHING FRACTIONS, INDETERMINATE EQUATIONS.

INEQUALITIES.

294. In addition to the axioms given on pages 16, 17, the student will find it advantageous to remember the following propositions:

I. *If the same quantity be added to or subtracted from two unequals, the sums or differences are unequal.*

Thus if $a > b$ then $a \pm c > b \pm c$.

II. *If two unequals be both multiplied, or both divided by the same positive quantity, the products are unequal, as also are the quotients.*

Thus, if $a > b$, $a - b$ is positive, and if m be positive then also $m(a - b)$ is positive, and $\therefore ma > mb$; similarly $\frac{1}{m}(a - b)$ is positive, $\therefore \frac{a}{m} > \frac{b}{m}$

III. *If the terms of an inequality be multiplied or divided by any negative quantity, or if the signs of all the terms be changed, the sign of inequality must be reversed.*

Thus, if $a > b$ then $a - b > 0$ or $-b > -a$, or $-a < -b$; so also if $a > b$ and $-m$ be any negative quantity, $a - b$ is positive $\therefore m(a - b)$ is negative, $\therefore m(b - a)$ is positive $\therefore mb > ma$ or $ma < mb$. Similarly $\frac{1}{m}(b - a)$ is pos. $\therefore \frac{b}{m} > \frac{a}{m}$ that is $\frac{a}{m} < \frac{b}{m}$

IV. *If any number of inequalities, all having the same sign of inequality, i.e. all $>$ or all $<$, be all multiplied together, left-hand members by left-hand members, and right by right, then the resulting products will form an inequality with the same sign.*

Thus, if $a > b$, $c > d$, $e > f$, then $ace > bdf$.

V. *If* a, b *and* n *be positive quantities, and* a > b, *then* $a^n > b^n$ *and* $\sqrt[n]{a} > \sqrt[n]{b}$.

Thus, $a > b$, ∴ last article, $a^2 > b^2$, ∴ $a^3 > b^3$, and so on· ∴ $a^n > b^n$; similarly $\sqrt[n]{a} > \sqrt[n]{b}$

VI. *If any number of inequalities having the same sign be added together, the sum is an inequality of the same kind.*

Thus, if $a > b$, $c > d$ and $e > f$, then $a + c + e > b + d + f$.

NOTE.—It does not, however, follow that if one inequality be subtracted from another, the difference is an inequality of the same kind. Thus, if $a > b$ and $c > d$ it does not always follow that $a - c > b - d$, since a may be nearer in magnitude to c than b to d; for example, although $7 > 5$ and $6 > 2$, $7 - 6$, is not greater than $5 - 2$, i. e. 1 is not greater than 3.

VII. *If the same quantity or two equal quantities be divided by each side of an inequality, the sign of inequality will be reversed.*

Thus $5 > 3$ but $\frac{15}{5} < \frac{15}{3}$, i.e. $3 < 5$; so also if $a > b$ then by dividing m by each we have $\frac{m}{a} < \frac{m}{b}$.

Ex. 1. Shew that if a be pos. and $b > a$ then $\frac{a-b}{a+b} > \frac{a^2-b^2}{a^2+b^2}$

Since $2 > 0$ multiplying by ab we have $2ab > 0$ ∴ also $a^2 + 2ab + b^2 > a^2 + b^2$ and dividing each by $(a^2 + b^2)(a + b)$ which is positive since a and b are both positive, we have $\frac{1}{a+b} < \frac{a+b}{a^2+b^2}$ and multiplying each of these by $a - b$ which is negative, because $b > a$ we have, proposition III, $\frac{a-b}{a+b} > \frac{a^2-b^2}{a^2+b^2}$.

Ex. 2. Shew that $x^2 + y^2 < \frac{x^6 + y^6}{x^4 - x^3y + x^2y^2 - xy^3 + y^4}$.

Because (Art. 134) $2xy < x^2 + y^2$, multiplying each each by xy we have $2x^2y^2 < x^3y + xy^3$,

And adding $x^4 - x^3y - x^2y^2 - xy^3 + y^4$ to each we have $x^4 - x^3y + x^2y^2 - xy^3 + y^4 < x^4 - x^2y^2 + y^4$ ∴ $1 < \frac{x^4 - x^2y^2 + y^4}{x^4 - x^3y + x^2y^2 - xy^3 + y^4}$ and multiplying each of these unequals by $x^2 + y^2$ we have

$$x^2 + y^2 < \frac{x^6 + y^6}{x^4 - x^3y + x^2y^2 - xy^3 + y^4}.$$

Ex. 3. Given $\left.\begin{array}{r}3x - 4 < x + 6 \\ 5x + 7 > 3x + 13\end{array}\right\}$ to find x in whole numbers.

From 1st inequality, $2x < 10$ $\therefore x < 5$. From 2nd inequality, $2x > 6$ $\therefore x > 3$ $\therefore x$ is > 3 and < 5, i.e. is any whole number between 3 and 5. Hence $x = 4$.

Exercise LXVI.

Find the limit to the value of x in the following inequations:

1. $7x - 13 < 22$.

2. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{6} + \frac{x}{12} - 7 > 9$.

3. $7x - 1 < 3x + 11$.

4. $2x + 5 > \frac{1}{2}x - 10$.

5. Given $\frac{ax}{5} + bx - ab > \frac{a^2}{5}$ and $\frac{bx}{7} - ax + ab < \frac{b^2}{7}$ to find the limits of x.

6. Prove that $a^3 + 1$ is equal to or greater than $a^2 + a$ according as $a = 1$ or $a > 1$.

7. Prove that $a^3 + 1 > a^2 + a$ when a is negative and numerically < 1.

8. Prove that $\frac{a}{b} + \frac{b}{a} > 2$ when a and b are both positive or both negative.

9. Given $\frac{1}{4}(x + 2) + \frac{1}{3}x < \frac{1}{2}(x - 4) + 3$ and $\frac{1}{4}(x + 2) + \frac{1}{3}x > \frac{1}{2}(x + 1) + \frac{1}{3}$ to find the value of x in whole numbers.

10. Shew that $a^2 + b^2 + c^2 > ab + ac + bc$ unless $a = b = c$.

11. Shew that $abc > (a + b - c)(a + c - b)(b + c - a)$ assuming that a, b and c are unequal.

12. Shew that $(1 + a + a^2)^2 < 3(1 + a^2 + a^4)$ unless $a = 1$.

13. Shew that $ab\,(a + b) + bc\,(b + c) + ca\,(c + a) > 6abc$ and $< 2(a^3 + b^3 + c^3)$ when a, b and c are positive quantities.

14. If $x^2 = a^2 + b^2$ and $y^2 = c^2 + d^2$ shew that $xy > ac + bd$.

15. If $a > b$ shew that $\sqrt{(a + b)(a - b)} + \sqrt{b(2a - b)} > a$.

16. Shew that $(a + b + c)^3 > 27abc$ and $< 9(a^3 + b^3 + c^3)$.

17. Prove that $(a + b)(b + c)(c + a) > 8abc$.

18. If x be real prove that $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ can have no value between 5 and 9.

19. Shew that $\frac{n^2 - n + 1}{n^2 + n + 1}$ lies between 3 and $\frac{1}{3}$ for all real values of n.

VANISHING FRACTIONS.

295. A vanishing fraction is one which assumes the form of $\frac{0}{0}$ when some particular value is given to some particular letter in both numerator and denominator.

Thus, $\frac{a^2 - b^2}{a - b}$ is a vanishing fraction when $b = a$, because then it becomes $= \frac{0}{0}$.

296. Now it will be readily seen that in the above example, and indeed in all others, the peculiarity arises from both numerator and denominator having a *common factor*, which factor $= 0$ under the assumed conditions. Thus, in the example given above we have $\frac{(a + b)(a - b)}{a - b}$, and striking out the common factor $a - b$ which $= 0$ when $b = a$ the expression becomes $a + b$ or $2a$ since $b = a$.

297. In order therefore to find the value of the fraction or more properly the *limit* to its value, we endeavour to find out the common factor involved, and casting it out, the result required is obtained by a simple reduction.

Ex. 1. Find the value of $\frac{x^4 - a^4}{x - a}$ when $x = a$.

OPERATION.

Here $\frac{x^4 - a^4}{x - a} = \frac{(x - a)(x + a)(x^2 + a^2)}{x - a} = (x + a)(x^2 + a^2.)$

Now making $x = a$ we have thus $= 2a \times 2a^2 = 4a^3$.

Ex. 2. Find the value of $\frac{x^m - a^m}{x - a}$ when $x = a$.

OPERATION.

Here $\frac{x^m - a^m}{x - a} = x^{m-1} + ax^{m-2} + a^2 x^{m-3} + a^3 x^{m-4} + \&c.,$ to m

terms and when $x = a$ this expression becomes $= a^{m-1} + a^{m-1} + a^{m-1} + a^{m-1} + \&c. \ldots$ to m terms $= ma^{m-1}$.

Ex. 3. Find the value of $\dfrac{x - a + \sqrt{2ax - 2a^2}}{\sqrt{x^2 - a^2}}$ when $x = a$.

OPERATION.

Here $\dfrac{x - a + \sqrt{2a(x-a)}}{\sqrt{(x-a)(x+a)}} = \dfrac{\sqrt{x-a}\left\{\sqrt{x-a} + \sqrt{2a}\right\}}{\sqrt{x-a}\,\sqrt{x+a}}$

$= \dfrac{\sqrt{x-a} + \sqrt{2a}}{\sqrt{x+a}} = \dfrac{\sqrt{a-a} + \sqrt{2a}}{\sqrt{a+a}} = \dfrac{\sqrt{2a}}{\sqrt{2a}} = 1.$

EXERCISE LXVII.

Evaluate the following vanishing fractions:

1. $\dfrac{1 - x^n}{1 - x}$ when $x = 1$.

2. $\dfrac{x^3 - a^3}{x^2 - a^2}$ when $x = a$.

3. $\dfrac{x - a^{\frac{1}{2}}x^{\frac{1}{2}}}{x - a}$ when $x = a$.

4. $\dfrac{x^2 + 2x - 35}{x^2 - 2x - 15}$ when $x = 5$.

5. $\dfrac{x^2 + \frac{5}{2}x - \frac{3}{2}}{x^2 - \frac{5}{2}x + 1}$ when $x = \frac{1}{2}$.

6. $\dfrac{x^3 + bx - ax^2 - ab}{x^2 - ax + b^2x - ab^2}$ when $x = a$.

7. $\dfrac{ax^2 + ac^2 - 2acx}{bx^2 - 2bcx + bc^2}$ when $x = c$.

8. $\dfrac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4}$ when $x = a$.

9. $\dfrac{x^3 + 2ax^2 - a^2x - 2a^3}{x^3 - 13a^2x + 12a^3}$ when $x = a$.

INDETERMINATE EQUATIONS.

298. It has been already stated, Art. 122, that when there are two or more unknown quantities involved in a single equation, the number of solutions is unlimited, and the equation is indeterminate.

Thus, $8x + 2y = 11$ is an indeterminate equation because the number of values which may be assigned to x and y is indefinite. This number may, however, be decreased : 1st by rejecting all fractional values; 2nd, by rejecting all negative values; 3rd, by rejecting all numbers that are squares or cubes, &c.

299. THEOREM I.—*The indeterminate equation* $ax \pm by = c$ *admits of at least one solution when* a *is prime to* b.

DEMONSTRATION.—$ax \pm by = c \therefore x = \dfrac{c \mp by}{a}$; and substituting in succession $0, 1, 2, 3 \ldots (a - 1)$ for y, a being prime to b, the several remainders must necessarily be different. For if any two values of y as v and v' give the same remainder r, q and q' being the quotients, then $c \pm bv = aq + r$ and $c \pm bv' = aq' + r$. Therefore $\pm bv \mp bv' = a(q - q')$, that is $b(v - v') = a(q - q')$ or $b(v' - v) = a(q - q')$; that is $b(v - v')$ and $b(v' - v)$ are divisible by a without a remainder. But by hypothesis b is prime to $a \therefore v - v'$ is divisible by a which is impossible, since v and v' are both by hypothesis less than a, and consequently $v - v'$ and $v' - v$ are less than a. Hence the remainders are all different and their number $= a$ and each is a positive integer less than a, consequently one of them must $= 0$, $\therefore x$ is an integral number for a certain integral value of y less than a, and these integral values of x and y satisfy the equation $ax \pm by = c$.

Ex. 1. Find integral values of x and y which satisfy the equation $5x + 23y = 170$.

SOLUTION.

Here $x = \dfrac{170 - 23y}{5}$ and substituting in succession 1, 2, 3, &c., for y and we find that 5 will do.

Thus, $\dfrac{170 - 115}{5} = \dfrac{55}{5} = 11 = x \therefore x = 11$ and $y = 5$.

300. THEOREM II.—*The equations* $ax \pm by = c$ *cannot be solved in positive integers if* a *and* b *have a divisor which does not also divide* c.

DEMONSTRATION.—For if it be possible let a and b have a common measure m which is not also a measure of c, and let a contain m, p times, and let b contain m, q times. Then $ax \pm by = c$ is

equivalent to $pmx \pm qmy = c$, or $px \pm qy = \frac{c}{m}$. And since both p and q are integers, and $\frac{c}{m}$ is a fraction, it follows that x and y cannot both be integral.

NOTE.—If a, b and c have a common measure the equation may be divided through by this, and thus a may be made prime to b. In the following articles this is always assumed to be done.

301. *Given one solution of the equation* ax ± by = c *in positive integers to find the general solution.*

Let $x = \beta$ and $y = \gamma$ be one solution of the equation $ax + by = c$.

Then $a\beta + b\gamma = c = ax + by \therefore a(\beta - x) = b(y - \gamma) \therefore \frac{a}{b} = \frac{y - \gamma}{\beta - x}$.

Now since $\frac{a}{b}$ is in its lowest terms, a being prime to b;

$\therefore$ whatever multiple $y - \gamma$ is of a the same multiple is $\beta - x$ of b. Let $y - \gamma = at$, then $\beta - x = bt$ where t is an integer, since we are only to obtain integral values.

Therefore $y = \gamma + at$ and $x = \beta - bt$ is the general solution.

Similarly writing $-b$ for b we obtain for the general solution of $ax - by = c$, $x = \beta + bt$ and $y = \gamma + at$.

Hence if one integral solution of the equation $ax \pm by = c$ can be detected, the others can be readily found by giving different integral values to t in the equations $x = \beta \mp bt$; $y = \gamma + at$.

Ex. 2. Given $3x + 4y = 39$ to find the positive integral values of x and y.

SOLUTION.

Here $x = 1$ and $y = 9$ is evidently one solution.

Then $x = 1 - 4t$ and $y = 9 + 3t$. Now let $t = -1$, then $x = 5$, $y = 6$, let $t = -2$ then $x = 9$, $y = 3$.

NOTE.—Since the values of x and y may be found by substituting for t in the general solution $x = \beta \mp bt$, $y = \gamma + at$, successively the values 0, ± 1, ± 2, ± 3, &c., it follows that the values of x and y taken in order constitute two arithmetical series, and consequently that as soon as two *contiguous* values of each are determined, the rest may be written at once.

302. THEOREM.—*The number of positive integral solutions is limited for* ax + by = c, *but unlimited for* ax − by = c.

DEMONSTRATION.—I. By Art. 301 it appears for $ax + by\ c =$ the general solution is $x = \beta - bt$ and $y = \gamma + at$ where $x = \beta$ and $y = \gamma$ is one solution and t is any integer positive or negative. Now since by hypothesis x and y are both to be positive, it is manifest that $\beta - bt$ must be positive, that is bt must be less than β, that is t is limited to integral values which are less than $\frac{\beta}{b}$. Hence the number of positive integral solutions of $ax + by = c$ is restricted.

II. Similarly in the general solution of $ax - by = c$ we have $x = \beta + bt$ and $y = \gamma + at$ where $x = \beta$, $y = \gamma$ is one solution and t is any integer positive or negative. Now since by hypothesis x and y are to be positive, $\beta + bt$ and $\gamma + at$ must be positive and since β, b and γ are positive it is manifest that t may be any negative integer such that $bt < \beta$ and $at < \gamma$ and that t may be any positive integer whatever. Therefore the number of positive integral solutions of $ax - by = c$ is unlimited.

303. In addition to the method indicated in Arts. 299, 301, for finding the values of the unknown quantities in an indeterminate equation, the following method may be studied with advantage.

Ex. 3. Solve $4x + 13y = 123$ in positive integers.

SOLUTION.

Divide by the least coefficient, which in this case is 4, then $x + 3y + \frac{y}{4} = 30 + \frac{3}{4}$. And since x and y are to be integral $x + 3y - 30$ is integral and $\therefore \frac{3-y}{4}$ which is the equal of $x + 3y - 30$ is integral.

Let $\frac{3-y}{4} = t$, an integer, then $3 - y = 4t \therefore y = 3 - 4t$.

Substitute this in the given equation for y, and $4x = 123 - 13\,(3 - 4t) \therefore x = \frac{123 - 39 + 52t}{4} = \frac{84 + 52t}{4} = 21 + 13t$.

Hence $\left.\begin{aligned} x &= 21 + 13t \\ y &= 3 - 4t \end{aligned}\right\}$

Take $t = 0$; then $x = 21 + 0 = 21$, $y = 3 - 0 = 3$.

Take $t = -1$; then $x = 21 - 13 = 8$, $y = 3 + 4 = 7$.

NOTE.—These are the only positive integral solutions, because as y is to be a positive integer, $3 - 4t$ must be a pos. int. $\therefore 4t < 3 \therefore t < \frac{3}{4}$ that is t may be any positive integer which is less than $\frac{3}{4}$, but 0 is the only positive integer less than $\frac{3}{4}$ $\therefore t$ cannot be a positive integer greater than 0. Similarly since x must be a positive integer $21 + 13t$ must be a pos. integ., i.e. t may be any negative integer which will not make $21 + 13t$ negative, i.e. $13t < 21$ or $t < \frac{21}{13}$, i.e. t when taken negatively must be an integer less than $\frac{21}{13}$ or in other words can only be -1.

Ex. 4. Solve $3x - 17y = 20$ in positive integers.

SOLUTION.

Divide by the least coefficient, 3. Then $x - 5y - \frac{2y}{3} = 6 + \frac{2}{3}$,

$\therefore \frac{2 + 2y}{3}$ is integral, $\therefore$ multiplying by 2, $\frac{4 + 4y}{3}$ is integral,

or $1 + y + \frac{1 + y}{3}$ is integral, $\therefore \frac{1 + y}{3}$ is integral $= t$, say,

Then $1 + y = 3t$ and $y = 3t - 1$. Substitute this in the given equation and $3x = 20 + 17(3t - 1) \therefore x = \frac{20 - 17 + 51t}{3} = 17t + 1$.

$$\text{Hence } \left.\begin{array}{l} x = 17t + 1 \\ y = 3t - 1 \end{array}\right\} \therefore \begin{array}{l} x = 18, 35, 52, 69, 86, \text{\&c.} \\ y = 2, 5, 8, 11, 14, \text{\&c.} \end{array}$$

According as $t = 1, 2, 3, 4, 5$, &c.

NOTE.—We multiply here by 2 in order to render the coefficient of y divisible by the denominator with a remainder 1, and this we seek to do in all cases.

Ex. 5. Solve in positive integers $5x + 19y = 207$.

SOLUTION.

Here dividing by 5 we have $x + 3y + \frac{4y}{5} = 41 + \frac{2}{5} \therefore \frac{4y - 2}{5}$ is integral, $\therefore$ multiplying by 4 we have $\frac{16y - 8}{5}$ integ., $\therefore 3y - 1 + \frac{y - 3}{5}$ is integ., $\therefore \frac{y - 3}{5}$ is integ. Let $\frac{y - 3}{5} = t$ then $y - 3 = 5t$ and $y = 5t + 3$. Substitute this value of y in the given equation and we get $5x = 207 - 19(5t + 3) = 207 - 57 - 19 \times 5t$.

$$\therefore \left.\begin{array}{l} x = 30 - 19t \\ y = 5t + 3 \end{array}\right\}$$

Now when $t = 0$ we have $x = 30$, $y = 3$.

When $t = 1$ we have $x = 11$, $y = 8$.

$\therefore$ The pos. int. solutions are $x = 30$ or 11 and $y = 3$ or 8.

Ex. 6. Solve in positive integers $41x + 68y = 2789$.

SOLUTION.

Dividing by 41 we have $x + y + \frac{27y}{41} = 68 + \frac{1}{41}$, $\therefore \frac{27y - 1}{41}$ is int.; multiplying by 3 we have $\frac{81y - 3}{41}$ int. $\therefore 2y - \frac{81y - 3}{41}$ is int.

$\therefore \frac{82y - 81y + 3}{41}$, that is $\frac{y + 3}{41}$ is int. Let $\frac{y + 3}{41} = t$ then $y = 41t - 3$.

Substitute this value of y in the given equation and

$$41x = 2789 - 68\ (41t - 3) = 2789 + 204 - 68 \times 41t.$$

$$\therefore x = \frac{2993 - 68 \times 41t}{41} = 73 - 68t.$$

Hence $\left.\begin{array}{l} x = 73 - 68t \\ y = 41t - 3 \end{array}\right\} \therefore \left.\begin{array}{l} x = 5 \\ y = 38. \end{array}\right\}$ when $t = 1$

It is evident that this is the only int. pos. solution, for $73 - 68t$ must be pos. int., so also must $41t - 3$ $\therefore 68t < 73$ or $t < \frac{73}{68}$; also $41t > 3$ or $t > \frac{3}{41}$ and the only positive integer between $\frac{73}{68}$, and $\frac{3}{41}$ is 1.

NOTE.—The student will not fail to observe the artifice made use of, in the 2nd line of the solution, to avoid using a large multiplier, and the trouble of searching for it, since it must be such as to render the coefficient of y divisible by 41 with a remainder 1.

Ex. 6. Given $\left.\begin{array}{l} 3x - 7y + z - 16 \\ 5x + 3y - 4z = -4 \end{array}\right\}$ to find the positive integral values of x, y, and z.

SOLUTION.

Multiplying the upper equation by 4 and adding the two together we have

$17x - 25y = 60$, and dividing by 17 we get $x - y - \frac{8y}{17} = 3 + \frac{9}{17}$

$\therefore \frac{8y + 9}{17}$ is integral.

So also is $\frac{16y + 18}{17}$, and so also is $y - \frac{16y + 18}{17}$ integral.

$\therefore \frac{y - 18}{17}$ is integral $= t$, say, then $y = 17t + 18$.

Then $17x = 60 + 25y = 60 + 25 \times 17t + 450 = 510 + 25 \times 17t$.

$x = 30 + 25t$ and $y = 17t + 18$.

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Hence $x = 5, 30, 55$, &c., and $y = 1, 18, 35$, &c.

But z also has to be positive and integral, and therefore the only values of x and y which are admissible are $x = 5$ and $y = 1$, and consequently $z = 8$.

Ex. 7. What is the least number which when divided by 4, 6 and 7 shall leave remainders 1, 3 and 5 ?

SOLUTION.

Let the number $= 4x + 1 = 6y + 3 = 7z + 5$. Then $4x - 6y = 2$.

$\therefore$ (I) $2x - 3y = 1 \therefore x - y - \frac{y}{2} = \frac{1}{2} \therefore \frac{y+1}{2}$ is int. $= m$, say

Then $y = 2m - 1$.

Also (II) $6y - 7z = 2$, that is $12m - 6 - 7z = 2 \therefore 12m - 7z = 8$

$\therefore m - z + \frac{5m}{7} = 1 + \frac{1}{7} \therefore \frac{5m-1}{7}$ is int. $\therefore \frac{15m-3}{7}$ is int.

$\therefore \frac{m-3}{7}$ is int. $= t$, say, then $m = 7t + 3$.

Hence $y = 2m - 1 = 14t + 6 - 1 = 14t + 5$.

$$x = \frac{6y+2}{4} = \frac{3}{2}y + \frac{1}{2} = \frac{42t+15}{2} + \frac{1}{2} = 21t + 8.$$

And $z = \frac{6y-2}{7} = \frac{84t+30-2}{7} = 12t + 4$.

Consequently $x = 8$, $y = 5$, and $z = 4$.

And the required number $= 4x + 1 = 33$.

Ex. 8. In how many ways can £80 be paid in sovereigns and guineas ?

SOLUTION.

Let x = number of sovereigns and y = number of guineas.

Then in shillings $20x + 21y = 1600 \therefore x + y + \frac{y}{20} = 80$.

$\therefore y = 20t$. And $20x = 1600 - 21y = 1600 - 21 \times 20t$.

$\therefore x = 80 - 21t$.

Then since $80 - 21t$ must be pos. and int. $\therefore$ 80 must be greater than $21t$, and since $21t < 80$, $t < \frac{80}{21}$ and $\therefore$ cannot exceed 3, and consequently there are only *three* ways of payment.

EXERCISE LXVIII.

Solve in positive integers.

1. $4x + 3y = 11$	2. $5x - 13y = 11$	3. $2x + 7y = 59$
4. $5x + 11y = 26$	5. $9x - 17y = 2$	6. $13x + 21y = 89$
7. $12x - 41y = -17$	8. $37x + 43y = 357$	9. $22x - 43y = 6$
10. $7x + 25y = 177$	11. $99x - 160y = 335$	12. $17x - 4y = 22$.

Find a positive integral solution of the following:

13. $\left.\begin{aligned} 2x + 3y + 4z &= 29 \\ 3x + 5y - 3z &= 9 \end{aligned}\right\}$ 14. $\left.\begin{aligned} 4x - 5y - 6z &= 17 \\ 2x + y + 11z &= 47 \end{aligned}\right\}$

15. In how many ways can the sum of \$697 be made up by bank notes of the respective value of \$3 and \$5?

16. In how many ways can \$27.30 be paid in twenty-five cent and ten cent pieces?

17. What is the simplest way for a person who has only guineas to pay £7 10s. 6d. to another who has only half crowns?

18. Find two integral square numbers whose sum is a square.

19. Find two integral square numbers whose difference is a square.

20. A basket of apples is known to contain between 90 and 100, and it is found that when they are counted four at a time, there are two over, and when counted six at a time there are also two over. How many are there in the basket?

21. Find the least integer which when divided by 6, 8 and 10 respectively shall leave remainders 1, 5 and 3.

22. How many fractions are there with denominators 10 and 15, whose sum is $\frac{23}{60}$?

23. A person bought 50 barrels of fruit, consisting of apples, pears, and cranberries, for \$250; the apples cost \$2 per barrel, the pears \$5 and the cranberries \$4, how many barrels were there of each?

24. How can a debt of £100 be paid with £5 notes, £1 note and crown pieces?

25. Divide 25 into two parts, one of which may be divisible by 2 and the other by 3.

26. Divide 24 into three such parts that if the first be multi-

plied by 36, the second by 24, and the third by 8, the sum of the three products may be 516.

27. Find a perfect number, *i.e.* one which is exactly equal to the sum of all its divisors.

28. What is the least odd integer which divided 10, 12, 14 shall leave remainders 7, 9 and 11 respectively?

29. A person buys 100 head of cattle of three different kinds for \$500. For the first he gives \$50 a head, for the second \$30, and for the third \$2, how many were there of each kind?

MISCELLANEOUS EXERCISES.

1. Simplify $\frac{1}{2}\{\frac{1}{2}(1-a)\} - \frac{1}{3}(\frac{1}{3}\{\frac{1}{3}(9a-6)\})$.

2. Prove that $(x^2+1-x^{-2})^2 - (x^2-1-x^{-2})^2 = 4(x^2-x^{-2})$.

3. Find the *G. C. M.* of $a^2+2ab+b^2$, a^3+b^3, a^4-b^2 and $x^3+2a^2b+2ab^2+b^3$.

4. Find the value of $\frac{x-b}{a} - \frac{x-a}{b}$ where $x = \frac{b^2}{b-a}$.

5. Given $x+y+z = 3(x+z-y) = 5(z-x-y) = 15$ to find the values of x, y and z.

6. Find the value of $5\sqrt[3]{135} - 3\sqrt[3]{40} + 2\sqrt[3]{625} - 4\sqrt[3]{320}$.

7. Given $x^4+1=0$ to find the values of x.

8. If $a:b::b:c$, and $b:c::c:d$, show that

$$a+b:b+c::b+c:c+d.$$

9. Shew that if $a:c::2a-b:2b-c$, then will a, $\frac{2}{3}b$ and $\frac{1}{2}c$ be in harmonic progression.

10. In the series $a + a\left(1-\frac{1}{p}\right)^{\frac{1}{n}} + a\left(1-\frac{1}{p}\right)^{\frac{2}{n}} + a\left(1-\frac{1}{p}\right)^{\frac{3}{n}} + \&c.$, the sum to infinity is p times, the sum of the first n terms.

11. Reduce $\dfrac{x^{\frac{3n}{2}} - x^{-\frac{3n}{2}}}{x^{\frac{n}{2}} - x^{-\frac{n}{2}}}$ and $\dfrac{x^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + a^{\frac{4}{3}}}{x^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}}$ to their simplest form.

12. Find the cube root of $343x^6 - 441x^5y + 777x^4y^2 - 531x^3y^3 + 444x^2y^4 - 144xy^5 + 64y^6$.

13. Simplify $x^{2m-n} \times x^{2n-p} \times x^{2p-m}$, and also $\frac{ab}{c} x^{p-q} \times \frac{bc}{a} x^{q-r} \times \frac{ca}{b} x^{r-p}$.

14. Find the product of $2x^2 + y + \frac{1}{2}x^{-2}y^2$ into $2x^2 - y + \frac{1}{2}x^{-2}y^2$; of $x^2 + ax + b^2$ into $x^2 - ax + b^2$, and of $x^m + y^p$ into $x^n + y^q$.

15. Simplify $\frac{3\sqrt{5} - 2\sqrt{3}}{3\sqrt{5} + 2\sqrt{3}} + \frac{2\sqrt{5} - 3\sqrt{3}}{3\sqrt{5} - 2\sqrt{3}}$.

16. Find the value of $\dfrac{1}{2x + \dfrac{1}{3x + \dfrac{1}{4x}}}$.

17. Find the value of

$$\frac{1}{4(2x-1)} - \frac{1}{4(2x+1)} + \frac{1}{2} \cdot \frac{2x+1}{(2x-1)(4x^2+1)}.$$

18. Find the values of x in the equations

(I) $\frac{a}{x+a} - \frac{c}{x+c} = \frac{a-c}{x+a-c}$.

(II) $\sqrt{(x-1)(x-2)} + \sqrt{(x-3)(x-4)} = 2$.

(III) $\frac{1}{x^2 - 2x - 15} + \frac{1}{x^2 + 2x - 35} - \frac{1}{x^2 - 13x - 48} = 0$.

19. If $n = \frac{b+c}{b-c}$, and b be the G. mean between a and c, then $\frac{a^2 - b^2}{a^2 + b^2}$ will be the H. mean between n and $\frac{1}{n}$.

20. A and B can together perform a piece of work in a days, which A and C can finish in b days, and B and C in c days. Find the time in which each can perform it separately.

21. Find the values of

$$\frac{a^2}{(a-b)(a-c)} - \frac{b^2}{(c-b)(b-a)} - \frac{c^2}{(b-c)(c-a)}.$$

22. Shew that $a^2 - \left(\frac{a^2 + 4b^2 - 9c^2}{4b}\right)^2 =$

$$\frac{(a+2b+3c)(a+2b-3c)(a-2b+3c)(2b-a+3c)}{16b^2}.$$

23. Find the two factors of $a^4 + b^4$, and the two factors of $a^4 - a^2b^2 + b^4$.

24. Simplify $\dfrac{x + y + \dfrac{x^2}{y}}{x + y + \dfrac{y^2}{x}}$.

25. Find the *l. c. m.* and also the *G. C. M.* of $x^2 + 3xy - 28y^2$, $x^2 - 2xy - 8y^2$ and $x^2 - 5xy + 4y^2$.

26. Find the general expression for the sum of a geometrical series when $r = \pm 1$.

27. If by the notation a_t we represent the tth term of a series; then in an *A*. series $(p - q)(a_m - a_n) - (m - n)(a_p - a_q)$ and in a *G*. series $\left(\dfrac{a_m}{a_n}\right)^{p - q} = \left(\dfrac{a_p}{a_q}\right)^{m - n}$. Required proof.

28. In comparing the rates of a watch and a clock, it was observed that one morning when it was 12h. by the clock, it was 11h. 59m. 49s. by the watch, and two mornings after when it was 9h. by the clock it was 8h. 59m. 58s. by the watch. The clock is known to gain one tenth of a second in 24 hours. Find the gaining rate of the watch.

29. Sum to 12 terms the series $8 + 12 + 18 +$ &c., and find the series both *A* and *G*, whose 3rd term is 4, and 6th term $\frac{32}{27}$.

30. The receiving reservoir at Yorkville is a rectangle 60 yds longer than it is broad, and its area is 5500 square yds. What are its dimensions?

31. Divide (I) $x^6 - 2x^3y^3 + y^6$ by $x^2 - 2xy + y^2$ by the method of factoring.

(II) $7x^8 + 5x^4 - 4x^3 + 3x + 9$ by $x^3 + 2x - 1$ by Horner's method.

(III) $x^m - x^{-m}$ by $x - x^{-1}$ to five terms. Also find the *rth* term, and if m be an even integer, prove that the complete quotient can be separated into two parts of which one is x^m times the other.

32. Find the square root of $37 + 20\sqrt{3}$, and of $4x + 2\sqrt{4x^2 - 1}$.

33. Find the fifth term of the expansion of $(a^4 - x^{-4})^{-3}$.

34. In how many ways can a party of seven men be formed out a company of 28 ?

35. Find the square root of $x^4 - 4x^2 + 9x^{-4} - 12x^{-2} \quad 10$ by inspection.

36. Find the three cube roots of unity, and show that their sum is equal to the sum of their squares.

37. Find the values of x and y in the equations:

$$\text{(I)} \quad \frac{x}{a} + \frac{y}{b} = 1 = \frac{x-a}{b} + \frac{y-b}{a}.$$

$$\text{(II)} \quad \left.\begin{aligned} x^2 &= 6x + 4y \\ y^2 &= 4x + 6y \end{aligned}\right\}$$

38. A and B sold 130 yards of calico, (of which 40 yards were A's and 90 yards B's) for \$42. Now A sold for \$1, one-third of a yard more than B sold for the same sum. How many yards did each sell for \$1?

39. Insert five $H.$ means between $\frac{1}{2}$ and $\frac{1}{7}$.

40. What is the difference between an identity and an equation, and to which of the two does

$$\frac{a+c}{(a-b)(x-a)} - \frac{b+c}{(a-b)(x-b)} = \frac{x+c}{(x-a)(x-b)} \text{ belong?}$$

41. Solve the equation $\sqrt[4]{x^2+1} + \sqrt{x} = 1$.

42. Simplify $ab - [(a+c)\,b - 3ac - \{ab - 2c\,(a-b)\}]$.

43. Simplify

$\frac{5}{6}x^2 - \frac{2}{3}xy - \frac{3}{10}y^2 - mx + ny \pm (\frac{3}{4}x^2 + xy - \frac{4}{15}y^2 + px - qy)$.

44. Reduce $\dfrac{(x^2 - 2x - 48)(x^2 + 3x - 28)}{(x^2 + 2x - 24)(x^2 - 3x - 40)}$ to its lowest terms.

45. Find the value of x in the equations

$$\text{(I)} \quad \frac{x}{(x-a)(x-b)} + \frac{a}{(a-b)(a-x)} + \frac{b}{(b-a)(b-x)} = \frac{1}{a-b}.$$

$$\text{(II)} \quad \tfrac{1}{4}\left(4 + \frac{3x}{2}\right) - \tfrac{1}{7}\left(2x - \tfrac{1}{3}\right) = \tfrac{31}{28}.$$

46. Find the value of x, y and z in the equations

$$x^2 + xy + y^2 = 37;\ y^2 + yz + z^2 = 28, \text{ and } z^2 + zx + x^2 = 19.$$

47. Find the least possible value of $2a^2 + 2a^2b + a^2b^2 - 2abx + b^2x^2$ for all real values of x.

48. Find the square root of $x^{6p} + 9x^{-6p} - 4x^{4p} + 4(x^{2p} - 3x^{-2p}) + 6$ by inspection.

49. Sum to 8 terms each of the series $3\frac{3}{7} + 6\frac{2}{7} + 9\frac{1}{7} +$ &c., and $81x^{12} - 54x^{10}y + 36x^8y^2 -$ &c. Also find the sum of the latter series *to infinity* when $x = 2y = 1$.

50. Find a geometrical series such that the sum of any three consecutive terms may be $\frac{1}{8}$ that of the succeeding six terms.

51. Simplify $x^{m(n-p)} . x^{n(p-m)} . x^{p(m-n)}$.

52. Reduce $\dfrac{x^4 + x^3 + 2x^2 + x + 1}{x^4 - x^3 + 2x^2 - x + 1}$ to its lowest terms.

53. Solve with respect to x the equation $x^2 - 2ax - 2bx - 3a^2 + 10ab - 3b^2 = 0$.

54. Given $\sqrt{y} - \sqrt{y - x} = \sqrt{20 - x}$, and also $\sqrt{y - x} : \sqrt{20 - x} :: 2 : 2$ to find the value of x and y.

55. Find by inspection the product of $(x^2 - 2x + 3)$ by $(x^2 + 2x + 1)$, and $(x^4 + 2x^2y^{\frac{3}{2}} + 3y^3)$ by $(x^4 - 2x^2y^{\frac{3}{2}} + y^3)$.

56. Solve the equation $x^3 + y^3 = a^3$, and $x^2y + xy^2 = b^3$.

57. A company at a tavern had \$35 to pay; but before the bill was paid, two of them left, and in consequence of this the remainder had each \$2 more to pay. How many were there in the company at first?

58. Find the ninth term in the expansion of $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^4$.

59. Find by inspection the coefficients of x^8 and x^{11} in the expansion of $(1 + ax - \frac{1}{2}ax^2 - 2a^2x^4 - x^5 + \frac{1}{2}ax^6 - 3ax^7)^2$?

60. Find two numbers such that the greater shall be to the less as their sum to a, and their difference to b.

61. Reduce $2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{x - 1}}}$ and also $\left(\dfrac{x^2 + 1}{x^2 - 1}\right)\left(\dfrac{x^2 + x^{-2} + 2}{x^3 + x^{-3}}\right)$ to simple quantities.

62. Find the value of the expression

$$\frac{x + 6}{x^2 + 2x - 35} + \frac{x - 4}{x^2 + 10x + 21} - \frac{x + 2}{x^2 - 2x - 15}.$$

63. Find the square root of $\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} - \left(\dfrac{x}{y} + \dfrac{y}{x}\right) + 2\frac{1}{4}$ by inspection, and also of $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.

64. Multiply $(x^m - 2y^n)$ by $(x^m - y^n)$, and also $(x^{m^2} + ax^m - b)$ by $(x^{m^2} - ax^m + b)$.

65. Divide $(12x^4 - 192)$ by $(3x - 6)$, and $(20a^4b^6 - 22a^3b^7 + 11a^2b^8 - 3ab^9)$, by $(4a^2b^3 - 2ab^4 + b^5)$. The former by factoring, and the latter by the method of detached coefficients.

66. If four quantities are in continued proportion, the first has to the fourth the triplicate ratio which it has to the second

67. Find the integral values of x which satisfy the *inequality* $x^2 < 10x - 16$.

68. Given $\dfrac{13 - 2\sqrt{x-5}}{13 + 2\sqrt{x-5}} = \frac{3}{23}$ to find the value of x.

69. If $\dfrac{a}{b}$ be any fraction whatever the sum of it and its reciprocal is greater than 2.

70. Shew that the sum of the cubes of any three consecutive numbers is divisible by three times the middle number.

71. Divide $(a^6 - 4a^4 + 7a^3 - 5a + 6)$ by $a^2 + 5a - 4$ synthetically. Also divide $(x^6 + x^{-6} - 2)$ by $(x^2 + x^{-2} - 2)$ by inspection.

72. Find the continued product of

$$\left\{ x^{\left(\frac{1}{2}\right)^{n-1}} - a^{\left(\frac{1}{2}\right)^{n-1}} \right\} \left(x^{\frac{1}{2}} + a^{\frac{1}{2}}\right)\left(x^{\frac{1}{4}} + a^{\frac{1}{4}}\right) \ldots. \text{ \&c., to } n \text{ factors.}$$

73. Simplify $\dfrac{13}{12(2x-3)} - \dfrac{7}{12(2x+3)} - \dfrac{x-4}{4x^2+9}$.

74. Find the product of $(\frac{1}{4}x^2 + \frac{1}{3}xy + \frac{2}{9}y^2)$ into $(\frac{1}{4}x^2 - \frac{1}{3}xy + \frac{2}{9}y^2)$, and of $\left(2x^{\frac{1}{4}} + 3y^{\frac{1}{4}}\right)\left(2x^{\frac{1}{4}} - 3y^{\frac{1}{4}}\right)\left(4x^{\frac{1}{2}} + 6x^{\frac{1}{4}}y^{\frac{1}{4}} + 9y^{\frac{1}{2}}\right)$ into the quantity $\left(4x^{\frac{1}{2}} - 6x^{\frac{1}{4}}y^{\frac{1}{4}} + 9y^{\frac{1}{2}}\right)$.

75. Given $2x\sqrt{3} - 3y\sqrt{2} = 6$ and $3x\sqrt{2} - 2y\sqrt{3} = 5\sqrt{6}$ to find the values of x and y.

76. Prove that if the series $1 + 3 + 5 + 7 +$ &c., be continued to any even number of terms, the sum of the latter half is three times the sum of the former half.

77. If the *A.* mean between two quantities be $\dfrac{a}{b} + \dfrac{b}{a} + 2$, and the *H.* mean be $\dfrac{a}{b} + \dfrac{b}{a} - 2$, then the *G.* mean will be $\dfrac{a}{b} - \dfrac{b}{a}$.

78. If a, b, c, be in *H.* progression, then will

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{b-a} + \frac{1}{b-c}.$$

79. If $r + s + t = v$, where r is constant and $s \propto \dfrac{x}{y}$ and $t \propto xy^2$, and when $x = y = 1$, $v = 0$, and when $x = y = 3$, $v = 8$, and when $x = 0$, $v = 1$, find v in terms of x and y.

80. Solve with respect to x the equations

$$\text{(I)}\quad \{(a+b)\,x+a-b\}\{(a+b)\,x+b-a\}=4ab.$$

$$\text{(II)}\quad \frac{ax}{b}-\frac{b}{a}=x+\frac{b}{ax}.$$

81. Find the continued product of $(a-b)(a+b)(a^2+b^2)+$ &c. to $n+1$ factors.

82. Divide $x^4-(a+b+p)x^3+(ap+bp-c+q)x^2-(aq+bq-cp)x-qc$ by x^2-px+q synthetically.

83. Find the square root of $a^2x^6+2abx^4+(b^2+2ac)x^2+c^2x^{-2}+2bc$.

84. Simplify

$$\left\{\frac{1}{(x+a)(x-b)}+\frac{1}{(x-a)(x+b)}\right\}\div\left\{\frac{1}{(x+a)(x+b)}+\frac{1}{(x-a)(x-b)}\right\}$$

85. Find the *G. C. M.* of $x^4+p^2x^2+p^4$ and $x^4+2px^3+p^2x^2-p^4$.

86. Find the *l. c. m.* of $2\frac{1}{2}(x^2+x-20)$, $3\frac{1}{3}(x^2-x-30)$ and $4\frac{1}{6}(x^2-10x+24)$.

87. Solve with respect to x the equation

$$(a^2-1)x^2-2(ab+1)x+b^2-1=0.$$

88. Simplify the following expression

$$\frac{x^3+x^{-3}+2(x+x^{-1})}{x^3-x^{-3}-2(x-x^{-1})}\cdot\left(\frac{x^2-1}{x^2+1}\right)^2$$

89. Prove that if to any square number there be added the square of half the number immediately preceding it, the sum will be a complete square; viz., the square of half the number immediately following it.

90. A cistern is furnished with two supply pipes A and B, and a discharge pipe C. If A and C be left open together for three hours, and C be then closed, the cistern will be filled in $\frac{1}{2}$ an hour more; if B and C be left open together for five hours and C be then closed, the cistern will be filled in $1\frac{3}{4}$ hours more; or it can be filled by leaving A open for $1\frac{2}{3}$ hours, and B $\frac{1}{2}$ hour. In what time can the cistern be filled or emptied by A, B, and C, separately.

91. Find the *G. C. M.* of $2x^5+2x^4-5x^3+4x^2-9$, and $3x^4+3x^3-10x^2-x+3$.

92. Find the *l. c. m.* of

$$apx^2+(aq+bp)\,x+bq, \text{ and } aqx^2-(ap-bq)\,x-bp.$$

$$\text{Also of } (x^2-xy);\ \left(x^{\frac{2}{3}}-y^{\frac{2}{3}}\right) \text{ and } (xy+y^2).$$

93. Solve the equations

$$\text{(I)}\quad \frac{x-2a}{3} = \frac{2x+6a}{7} - \frac{x+2a}{13}$$

$$\text{(II)}\quad \frac{x-1}{2} - \frac{x+1}{3} = \frac{3}{x+1} - \frac{2}{x-1}$$

$$\text{(III)}\quad \sqrt{x+4} + \sqrt{2x+6} = \sqrt{3x+34}$$

$$\text{(IV)}\quad x^2(y-1) + 3y(x^2-1) = \sqrt{x^2+3y} \text{ and } x^2y = 5$$

94. Form the equation whose roots are 2, 3 and $-2 \pm \sqrt{-3}$

95. Simplify $a - (a - m) - \{-(-\{-a-(-m-\{-(m-a)\})\})\}$

96. Resolve $a^8 + b^8$ into its component factors.

97. If $A.$, $G.$ and $H.$ be the arith., geom, and harm. means between two quantities a and b, then will

$$\frac{H}{A} = 1 + \frac{(H-a)(H-b)}{G^2}.$$

98. Find the time between two successive transits of the minute-hand over the hour-hand of a common clock.

99. The opposite sides of a rectangle are each increased by a units in length, and the other two sides decreased by b units, and the area is found to be unaltered; but if these changes in the sides had been respectively c and d units, the area would have been diminished by e square units. Find the sides and examine the nature of the problem when $ad = bc$, and $bc + e = cd$

100. Given $\left(\dfrac{x-a}{x-b}\right)^3 = \dfrac{x-2a+b}{x-2b+a}$, to find x.

101. Divide $5x^5 - 3x^2 + 1$ by $x^2 - 2x + 3$ by Horner's method, exhibiting both the complete remainder, and the continuation of the quotient in descending powers of x.

102. Find the G. C. M. of $x^2y + xy^2 - 3x^2 + 3y^2 - 9x + 9y - 2y^3$, and $x^2y + 2xy^2 + x^2 + 4xy - 5y^2 + 2x - 2y - 3y^3$, and examine what the result becomes when $y = 1$.

103. If $a \propto \sqrt{b}$ and $c^2 \propto b^3$ shew that $ac \propto b^2$.

104. Resolve $a^{12} + m^{12}$ into *four* elementary factors.

105. Reduce $\dfrac{m^2-(p-q)^2}{(m+q)^2-p^2} + \dfrac{p^2-(q-m)^2}{(m+p)^2-q^2} + \dfrac{q^2-(m-p)^2}{(p+q)^2-m^2}$ to its simplest form.

106. Given $2^{x+1} + 4^x = 80$ to find x.

107. If x be real, prove that $x^2 - 8x + 22$ can never be less than 6.

108. If $a \propto d^2$, $b^3 \propto d^4$ and $c^3 \propto$ inversely as d, shew that the product abc varies as if each of the three varied directly as d.

109. Shew that the sum of n consecutive odd numbers beginning with $2m + 1$ exceeds the sum of the first n odd numbers beginning with unity by twice the product of m and n.

110. If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio $m:n$, shew that $\frac{b^2}{ac} = \frac{(m+n)^2}{mn}$.

111. Prove that $\frac{a^4(b^2-c^2) + b^4(c^2-a^2) + c^4(a^2-b^2)}{a^2(b-c) + b^2(c-a) + c^2(a-b)} =$ $(a+b)(b+c)(a+c)$

112. Every square number is either divisible by 3 or becomes so by the addition of 2, and the product of any three consecutive integers, the middle one of which is odd, is divisible by 24.

113. Prove that $\{n(n+1)\}^2 - \{(n-1)n\}^2 = 4n^3$.

114. Find the value of $\frac{(ab+1)(x^2+1)}{(xy+1)(a^2+1)} - \frac{x+1}{y+1}$ when $x = \frac{1+a}{1-a}$ and $y = \frac{1+b}{1-b}$

115. Divide synthetically $7x^5 + 21x^4y + 35x^3y^2 + 35x^2y^3 + 21xy^4 + 7y^5$ by $x + y$, and the result by $x^2 + xy + y^2$.

116. Employ the method of detached coefficients to find the G. C. M. of $18x^4 + 9x^3 - 17x^2 - 4x + 4$, and $8x^4 + 4x^3 - 6x^2 - x + 1$.

117. Resolve the quantics given in the last question into their elementary factors.

118. Reduce to a single fraction

$$\frac{3}{4(1-x)^2} + \frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x^2)}$$

119. Is the following expression an identity or an equation

$$\left(x + \frac{5a}{2}\right)\left(x - \frac{3a}{2}\right) + ax = (x+5a)(x-3a) + 11\tfrac{1}{4}\,?$$

If $a = 1$, how then?

120. If a, b and c be in *H.* progression then will $\frac{ab}{a+b}, \frac{ac}{a+c}$ and $\frac{bc}{b+c}$ also be in *H.* progression and $\frac{b+c}{a}$, $\frac{c+a}{b}$ and $\frac{a+b}{c}$ will be in *A.* progression.

121. If $a \propto b$ and $c \propto d$ then will $ad \propto bc$.

122. If there are two circles each of radius 3, and four others of radii 4, 5, 6, and 7 respectively, shew that they can all be made into a single circle of radius 12, assuming that the area of a circle varies as the square of its radius.

123. Given the first term of an *A.* series = 11, and that the sum of the first 3 terms = the sum of the first 9 terms, to find the series.

124. Given any two terms of a *G.* series to construct it.

125. Find the *G.* series whose 1st term = 3, 5th term = $\frac{16}{27}$, and sum of first five terms = $2\frac{1}{27}$.

126. Prove that the latter half of $2n$ terms of an *A.* series is one-third of the sum of $3n$ terms of the same series.

127. If S^1 denote the sum of n terms of the series $1 + 5 + 9 +$ &c. and S_2 denote the sum to $(n - 1)$ or to n terms of the series $3 + 7 + 11 +$ &c., prove that $S_1 + S_2 = (S_1 - S_2)^2$.

128. Find the 7th, the 10th and the general term in the expansion of $(1 + x^{-2})^{-\frac{2}{3}}$.

129. Form the equation whose roots are $1, -1, 2, -2$ and $3 \pm \sqrt{-2}$.

130. Assuming that -1, 1 and 1 are three roots of the equation $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$ to find the other two roots.

131. Find what quantity must be added to each term of the ratio $a : b$ in order to make it four times as great as the ratio $c : d$.

132. Shew that $\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{\frac{1}{4}} = \frac{\sqrt{2}}{1+\sqrt{3}}$

133. Given $\left.\begin{array}{r} 2x + 3y + 4z = 29 \\ x - 2y + 3z = 8 \end{array}\right\}$ to find x, y, z in positive integers.

134. Find the value of the vanishing fraction $\frac{x^{n+1} - y^{n+1}}{x^n y^n (x-y)}$ when $x = y$.

135. The sum of two numbers is 45, and their *l. c. m.* is 168, what are the numbers?

136. Given $\frac{1}{5}\left\{\frac{(x+1)(x-3)}{(x+2)(x-4)}\right\}+\frac{1}{9}\cdot\frac{(x+3)(x-5)}{(x+4)(x-6)}-\frac{2}{13}\cdot\frac{(x+5)(x-7)}{(x+6)(x-8)}=\frac{92}{585}$ to find x.

137. Given $\left.\begin{array}{c} 2xy-4y^2+x^2=4 \\ x^2-y^2=36 \end{array}\right\}$ to find the values of x and y.

138. Prove that the fraction $\frac{1}{81}$ on being converted into a decimal will continually produce, successively in order, the digits 0, 1, 2 . . . . 9 inclusive with the exception of 8.

139. Prove that the roots of $ax^2-bx=a^2x-ab$ are rational.

140. Solve the equation $(a+x)(b+x)=nab$.

141. Find the value of x in the equation $1+\sqrt{x}=6x$.

142. Given $\sqrt{x}+\sqrt{x-1}=\sqrt{x+1}$ to find x.

143. Solve with respect to x, y and z the equations

$$x+y+z=\frac{a^2}{x}=\frac{b^2}{y}=\frac{c^2}{z}$$

144. If a number be multiplied by 4, and the same number reversed be multiplied by 5, the sum of the products is exactly divisible by 9.

Prove this, and infer the general proposition of which it is a particular case.

145. Simplify $(a+b)(b+c)-(a+1)(c+1)-(a+c)(b-1)$.

146. Find, without actually multiplying, the product of

$$\left(\frac{x^2y^2}{9}-xy+9\right)\text{ into }\left(\frac{xy}{3}+3\right)$$

147. Find, without actually dividing, the quotient of $(ax+by)^2+(cx+dy)^2+(ay-bx)^2+(cy-dx)^2$ by x^2+y^2.

148. Extract the square root of $a^2(x^2+4)-2a(x+2)+4a^2x+1$ by inspection.

149. Find the G. C. M. of $a^2+b^2-c^2+2ab$, and $a^2-b^2-c^2+2bc$ by factoring.

150. Divide synthetically $4x^4+5x^2+1$ by x^3+2x-1 obtaining the exact remainder, and also four terms of the remainder expressed in descending powers of x.

151. Expand $\frac{1}{1-x+x^2}$ in ascending powers of x.

152. Simplify $\left(\frac{a}{a+b}+\frac{b}{a-b}\right) \times \left(\frac{a}{a-b}-\frac{b}{a+b}\right)$

153. Divide $\left(\frac{a}{a+c}-\frac{b}{b+c}\right)$ by $\left(\frac{c}{b+c}-\frac{c}{a+c}\right)$

154. Reduce to a single fraction in its lowest terms

$$\frac{3(x-2)}{(x-1)(x-3)}-\frac{1}{x-1}-\frac{1}{(x-2)}-\frac{1}{x-3}$$

155. Prove that

$$\frac{(xy+1+2x)(xy+1+2y)+(x-y)^2}{x^2y^2+1-x^2-y^2}=\frac{(x+1)(y+1)}{(x-1)(y-1)}$$

156. Find the conditions necessary in order that the equations $ax^2+bx+c=0$ and $a_1x^2+b_1x+c_1=0$ may have

(I) One root common.

(II) Roots equal in magnitude, but of contrary signs.

157. Solve the equation $\frac{x+1}{2}-\frac{2x-1}{3}=\frac{3x+4}{4}-\frac{5x-6}{3}$

158. Given $\frac{(x-1)(x+4)}{(x+3)}=\frac{(3+x)(2-x)}{1-x}$ to find the value of x.

159. Find the value of x in the equation

$$\frac{1+2x}{1-2x}=\frac{1+x+\sqrt{1+2x}}{1-x-\sqrt{1-2x}}$$

160. Find x in the equation

$$\frac{(x-1)^2\,(n-1)^2+4n}{(x+1)^2\,(n-1)^2+4n}=\text{P}.$$

and shew that if n be positive and x real, the value of the left hand members always lies between n and $\frac{1}{n}$

161. Find the *A.*, *G.* and *H.* means between $\frac{3}{4}$ and $\frac{4}{3}$.

162. If *H.* be the harmonic mean between a and b, prove that it is also the *H.* mean between $(H-a)$ and $(H-b)$

163. Find the 37th term of the series $6+\frac{35}{6}+\frac{17}{3}+$ &c., and also the sum of the sums of the first 31 terms and 42 terms.

164. Find the sum of n terms of the series $3\frac{1}{3}+2+1\frac{1}{5}+$ &c.

165. Find the sum of n terms of the series $1-0\cdot4+0\cdot16-0\cdot64+$ &c., and also the difference between the sum to infinity and the sum to n terms.

166. There are p arithmetical series, each continued to n terms; their first terms are the natural numbers 1, 2, 3, &c., and their common differences are the successive odd numbers 1, 3, 5, &c. Prove that the sum of all of them is the same as if there were n such series each continued to p terms.

167. Find the continued product of $x - \sqrt{xy} + y$, $x + \sqrt{xy} + y$ and $x^2 - xy + y^2$.

168. Find the value of $y\,(x^2 - 3y)^{\frac{1}{2}} + x\,(x^2 + 3y)^{\frac{1}{2}}$ when $x = 5$ and $y = 8$.

169. Extract the 4th root of $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$.

170. If $a : b :: c : d$ shew that

$$\frac{1}{a} - \frac{1}{2b} - \frac{1}{3c} + \frac{1}{4d} = \frac{1}{ad}\left(\frac{a}{4} - \frac{b}{3} - \frac{c}{2} + d\right)$$

171. Solve the equation $\dfrac{2x+3}{x+1} = \dfrac{4x+5}{4x+4} + \dfrac{3x+3}{3x+1}$ giving the rule and reason for each step of the operation.

172. Solve with respect to x the equation

$$\frac{1}{x} + \frac{1}{x+b} = \frac{1}{a} + \frac{1}{a+b}$$

173. When $x = \dfrac{a+1}{ab+1}$ and $y = \dfrac{ab+a}{ab+1}$ reduce $\dfrac{x+y-1}{x+y+1}$ to its lowest terms.

174. Shew that $2(x-y)(x-z) + 2(y-z)(y-x) + 2(z-x)(z-y)$ can be resolved into the sum of three squares.

175. Divide $a^4 + b^4 - c^4 - 2a^2b^2 + 4abc^2$ by $(a+b)^2 - c^2$.

176. Find the G. C. M. of $x^8 - 1$ and $x^{10} + x^9 + x^8 + 2x^7 + 2x^4 + 2x^3 + x^2 + x + 1$.

177. Reduce $\dfrac{2x+3}{(x+5)(x+1)} - \dfrac{x+2}{x^2+1} - \dfrac{x-7}{(x+5)(x-1)}$ to a simple quantity.

178. Reduce $\dfrac{a + b\sqrt{-1}}{a - b\sqrt{-1}} + \dfrac{a - b\sqrt{-1}}{a + b\sqrt{-1}}$ to a simple quantity.

179. If four positive quantities be in *A.* Progression, the sum of the extremes is equal to the sum of the means; but if in *G.* or *H.* Progression the former sum is the greater. Required proof.

180. Shew that in an ascending A. series if the least term be the common difference, the sum of $(2n - 1)$ terms is n times the greatest term.

181. Solve with respect to x the equation $\frac{a}{x} + \frac{\sqrt{a^2 - x^2}}{x} = \frac{x}{b}$.

182. Given $3x^{\frac{5}{3}} + x^{\frac{5}{6}} = 3104$ to find the values of x.

183. Find the value of x in the equation

$$\frac{x+a}{x-a} - \frac{x-a}{x+a} = \frac{b+x}{b-x} - \frac{b-x}{b+x}$$

184. Given $x + \sqrt{\{x^2 + \sqrt{x^2 + 96}\}} = 11$ to find the values of x.

185. Find a number of two digits, such that when divided by the difference of the digits, the quotient is 21; and when divided by the sum of the digits and the quotient increased by 17, the digits are inverted.

186. Two horses A and B, trot twice round a course two miles long. B passes the post the first time $2'$ before A, but in the second round A increases and B slackens his pace by 2 miles per hour, and A does the round in $2'$ less than B. Find their rates and which horse wins.

187. With any five consecutive integers, the continued product of the first, middle, and last, added to the cubes of the other two is equal to the product of the middle number by the sum of the squares of the middle three. Required proof.

188. Prove that $x^4 + y^4 + (x+y)^4 = 2(x^2 + xy + y^2)^2$.

189. Multiply $x^3 + y^3 + x^2y + xy^2$ by $x^3 - y^3 - x^2y + xy^2$.

190. Find the value of $ax^2 - \frac{1}{4}x^4$ when $x = (a+b)^{\frac{1}{2}} \pm (a-b)^{\frac{1}{2}}$.

191. Divide $ax^3 + 2cxyz + by^3 + ax^2(y+z) + by^2(x+z) + 2cxy(x+y)$ by $x+y+z$, synthetically.

192. What is the quotient of $x^{n^2} - 1$ divided by $x^n - 1$.

193. Simplify $1 - \{1 - (1-x)\} + 2x - (3 - 5x) + 2 - (-4 + 5x)$.

194. Express $a(b+c)^2 + b(c+a)^2 + c(a+b)^2 - \{(a-b)(a-c)(b+c) + (b-c)(b-a)(c+a) + (c-a)(c-b)(a+b)\}$ in its simplest form.

195. Express in the simplest form the sum of

$$(b+c-a)x + (c+a-b)y + (a+b-c)z$$
$$(c+a-b)x + (a+b-c)y + (b+c-a)z$$
$$(a+b-c)x + (b+c-a)y + (c+a-b)z$$

196. Find the product of $(x^3 + 6x^2y + 12xy^2 + 8y^3)$ by $(x^3 - 6x^2y + 12xy^2 - 8y^3)$ also of $(a + b\sqrt{-1})(a - b\sqrt{-1})$.

197. Find the value of $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$ Also the product of $(x^2 + 1 + x^{-1})$ by $(x^2 - 1 + x^{-1})$.

198. Divide $(2x^4 - 3x^3y + 4x^2y^2 - 5xy^3 + 6y^4)$ by $6x^2y^2$; and also $(x^4 + 4x + 3)$ by $(x^2 + 2x + 1)$.

199. Find by inspection the quotient of $(8x - y^3) \div (x^{\frac{1}{3}} - \frac{1}{2}y)$ and of $(x^3 - apx^2 + a^2px - a^3) \div (x - a)$.

200. Find by factoring the G. C. M. of

(I) $x^2 - 3x - 4$, $x^2 - 2x - 8$ and $x^2 + x - 20$.

(II) $3x^3 + 4x^2 - 3x - 4$ and $2x^4 - 7x^2 + 5$

(III) $(x^m + a^m)(x^n - a^n)$ and $(x^n + a^n)(x^m - a^m)$.

201. Find the *l. c. m.* of

(I) $x^2 - ax - 2a^2$, $x^3 + ax^2$ and $ax^2 - a^3$

(II) $x^3 - x^2y - a^2x + a^2y$ and $x^3 + ax^2 - xy^2 - ay^2$

202. Find the value of

$$\frac{(a+b-c)^2 - d^2}{(a+b)^2 - (c+d)^2} + \frac{(b+c-a)^2 - d^2}{(b+c)^2 - (a+d)^2} + \frac{(c+a-b)^2 - d^2}{(c+a)^2 - (b+d)^2}$$

203. Reduce $\dfrac{x^2 + y^2 - z^2 + 2xy}{x^2 - y^2 - z^2 + 2yz}$ to its lowest terms.

204. Simplify the expression $\dfrac{a^3 + a^2b}{a^2b - b^3} - \dfrac{a(a-b)}{(a+b)b} - \dfrac{2ab}{a^2 - b^2}$

205. Reduce $\left(a + \dfrac{ax}{a-x}\right)\left(a - \dfrac{ax}{a+x}\right) \div \left(\dfrac{a+x}{a-x} + \dfrac{a-x}{a+x}\right)$ to a simple quantity.

206. Find the value of $\dfrac{x+2a}{x-2a} + \dfrac{x+2b}{x-2b}$ when $x = \dfrac{4ab}{a+b}$

207. Find by inspection the square roots of

(I) $x^4 - 4x^3 + 8x + 4$

(II) $4x^{4n} - \frac{4}{3}x^{5n} + \frac{1}{9}x^{6n}$

(III) $\dfrac{a^2}{b^2} + \dfrac{b^2}{c^2} + \dfrac{c^2}{a^2} - 2\dfrac{a}{c} - 2\dfrac{c}{b} + 2\dfrac{b}{a}$

208. If $a^2x^2 + bx + bc + b^2$ be a perfect square, shew that $\dfrac{1}{4a^2} = \dfrac{c}{b} \cdot 1$

209. Solve with respect to x the equations

(I) $mnx + amn = n^2x + am^2$

(II) $\dfrac{8 - x}{2} - \dfrac{2x - 11}{x - 3} = \dfrac{x - 2}{6}$

210. Find the values of x in the equations

(I) $\dfrac{7x + 1}{6\frac{1}{2} - 3x} = \dfrac{80}{\cdot 3}\left(\dfrac{x - \frac{1}{2}}{x - \frac{2}{3}}\right)$

(II) $x^2 - 2ax - 2bx - 3a^2 + 10ab - 3b^2 = 0$

211. Find the values of x, y and z which satisfy the equations

(I) $\dfrac{x - ay}{b} = 1 = \dfrac{ax + y}{c}$

(II) $x^2 + xy + y^2 = 37$ and $x + y = 7$.

212. Solve the simultaneous equation

$z(x + y) = a^2 + b^2$; $x(y + z) = b^2 + c^2$; $y(z + x) = c^2 + a^2$.

213. The difference between the ages of A and B is twice as great as the difference between the ages of B and C, and the sum of the ages of A and B is half as much again as the age of C; six years ago it was only one-third more. Find their ages.

214. Sum the following series:

(I) $1\frac{1}{2} + 3 + 4\frac{1}{2}$ to 12 terms.

(II) $1\frac{2}{3} + 2\frac{4}{9} + 3\frac{8}{27}$ to n terms.

(III) $\sqrt{2} + \frac{2}{3}\sqrt{3} + \frac{2}{3}\sqrt{2}$ to infinity.

215. If $a_1 . a_2 . a_3 \ldots . a_t = a_1^{t^2}$ then will

$a_1 + a_2 + a_3 + \ldots . a_n = a_1 . \dfrac{a_1^{2n} - 1}{a_1^2 - 1}$. Required proof.

216. Given $(x + 5)(x + 1) = 4\sqrt{2x + 1}\,(x - 1)$ to find x.

217. Find the value of x in the equation

$$(3x - 4)(5x - 1)(1 - 2x^2) = 4.$$

218. Find to $4n$, $4n + 1$, $4n + 2$ and $4n + 3$ terms the sum of the following series

$$1 + 1 + 2 - 2 + 3 + 4 + 4 - 8 + 5 + 16 + \&c.$$

219. The number of matches in the side of a certain rectangular bunch is > 10 but < 20, while the number in the end is < 10. When the digit expressing the number in the end is written to the left of the expression for the number in the side, the number

so formed is to the whole number of matches in the bunch as a certain number a is to 2; but if this digit is written to the right of the expression for the number in the side, the number thus formed is the whole number of matches as $a - 10 : 4$. Also a second bunch similar in form to the first, and containing as many matches in its perimeter as there are matches in the first bunch, contains four times as many matches as the the first bunch. Find the whole number of matches in the bunch.

220. Shew that, in the preceding problem, if the last condition had not been given, the solution found above would have been the only integral solution of the problem.

221. A person travels by railway from Stratford to Toronto and back. In coming down he finds that when he travels by express he is as many hours on the way as his fare is cents per mile, but when he travels by the accommodation train he is half as many hours on the way as there are units in the square of the number of cents in his fare per mile, the fare being the same by both trains. In returning, the express by which he travels goes slower than the express by which he came down by an average (including stoppages in both cases) of as many miles per hour as there are cents in his fare per mile, the fare being the same as in coming down. He now calculates that if the fare had varied as the speed of the trains, he would have gained a cent a mile by taking the accommodation train to Toronto—the fare on the express to Toronto remaining the same—and in returning he would have gained as many cents as there were miles in the average speed (including stoppages) of the train. Find the distance from Toronto to Stratford, and the fare between them.

222. Given $\sqrt{x^2+25}\,\{x^2(x^2+9)(\sqrt{x^2+25}-1)-45\} = 5x^2+225$ to find the values of x.

223. Two persons engage to dig a trench 100 yds. long for \$100, but one end being more difficult to dig than the other it is agreed that the one digging the harder end shall receive \$1·25 per yard, while the other receives but \$0·75 per yard. At the termination of the job it is found that they each receive \$50. How many yds. did each dig?

Shew algebraically that this problem is impossible.

224. A square and a rectangle are (I) equal in area, (II) equal in perimeter. The number of square inches in the area of the square is m times the number of linear units in its perimeter, and the number of square units in the area of the rectangle is n times the number of linear units in its perimeter. Find the length of the sides of the rectangle.

225. Two boys find upon trial that the distances to which they can respectively throw a stone are in proportion to their ages, and that the throw of the elder is 24 feet longer than that of the younger. After the lapse of a year they try again with the same stone and find that the elder can throw it but 22 feet farther than the younger, and that the gain of each is in the same ratio to the age of the other. Also the *H.* mean between their ages at the latter trial is equal to the quotient obtained by dividing the length of the longest throw made by the difference between the *A.* mean of the 1st throws and that of the 2nd throws; and if the antecedent of the ratio compounded of the ratio of the throw of one to his age in the first instance and the ratio of his gain to the age of the other on the second trial, be multiplied by $\frac{1}{4}$ of the product of their ages on the second trial the ratio of which the resulting ratio is the duplicate, will be the the same as the ratio compounded of the ratio of the throw of one to his age at the first trial, and the reciprocal of the ratio of his gain to the age of the other at the second trial. Find their ages and the distance to which they throw the stone.

ANSWERS TO EXERCISES.

Exercise IV.

1. 0	2. 18	3. 14	4. 2
5. 3	6. 0	7. 48	8. 16
9. 48	10. 0	11. 24	12. 2700
14. $5 < 6$	15. each = 0	16. $6 > 5$	17. each = 10
18. each = 2	19. 2	20. 44	21. 19
22. -112	23. -3	24. 22	25. 8

Exercise V.

1. $43a$.
2. $-26ab^2$.
3. $19(a + b - c^2)$.
4. $27a(x - y^2)^{\frac{1}{3}}$.
5. $27a - 13y + 23$.
6. $16(x + y) + 28a - 20abc$.

15\. 7. $8(a + b)x - 19(c + d)y - 23(d + f)z$.

8. $15a^2b^3x^{\frac{2}{3}} + 12a^3b^2x^{\frac{2}{3}} - 13a^2b^{\frac{2}{3}}x^3 - 17a^3b^{\frac{2}{3}}x^2$.

Exercise VI.

1. $3a + 3c$; $a + 3c$; $4a + 4b - 7c$.
2. $8ab - 7ay + 13cd$.
3. $-a^2x^{\frac{2}{3}} - 7(a + b) - 12x^{\frac{2}{3}}y - 20$.
4. $2a - 2b$.
5. $5xy + 14ab + 17$.
6. $5 + 8a - 5b + 8c$.
7. $6ab + 6xy - 5cd - m + 16c$.
8. $17 - 25m^2x + 20xy$.
9. $m^{\frac{1}{2}}n^{\frac{2}{3}}$.
10. $18\sqrt{a} - 8\sqrt[3]{3} + 14\sqrt[4]{4} + 6\sqrt[5]{a} + 19\sqrt[6]{c}$.
11. $20xy - 10ay + 2\sqrt{x} + 25\sqrt[3]{y}$.
12. $4(ax + by - cz)^{\frac{1}{4}} + 12\sqrt{m + n} + 16(x - y)$.

Exercise VII.

1. $a + b + c + m + 3p + x + y$.
2. $-3xz - 5c^3$
3. $7c + 4x^2 + 2(x - y)$.
4. $10x^2y - a^2b + 7$
5. $6a + 15b + 5ab - 3m^2n + 5x + y$.
6. $6\sqrt{x} - 5\sqrt{a + y} + 18$
7. $4x^3 - 2y^3 + 3y^2 + 2y$.
8. $3\sqrt[3]{xz + xy - yz} + am - 7a^2y + x^2y - m^3$

Exercise VIII.

1. $a^2y^2z - 11xy^3 + 11az^2 + 4xy + 20m.$
2. $14a - 14c - 13\sqrt{a - b^2} + 4xy^2 + m^2.$
3. $2cd - 3(a + b)\sqrt[3]{x^2 - y}.$
4. $4(xy + y^2 - z^3)^{\frac{1}{5}} + 14a^{\frac{1}{3}}x^{\frac{1}{4}} - 14\sqrt[3]{m}.$
5. $16 + 7\sqrt[4]{8} - 23y - 9\sqrt{a - b}.$
6. $6m - 2c - 11e - 25x + 12y + abcd.$

Exercise IX.

1. $14 - m - 5c - e.$	2. $2a - 2b - 2c.$	3. $x - 5a - 2$
4. $6 + m.$	5. $11a - 3c - 5d + m$	6. $2a^2 - c^2 - m^2.$
7. 2	8. $5a^2 + 7x + 3m^2 + 2x^2$	9. $8a^2bc - 2m.$
10. $a + 1$	11. $a - 8b - 6c.$	12. $-a - 5am - 2c - 17$

Exercise X.

1. $(a - b) + (c - d) - (e - m) - (f + r) - (s - v) + (w + x)$
2. $(a - b + c) - (d + e - m) - (f + r + s) + (v + w + x)$
3. $(a - b + c - d) - (e - m + f + r) - (s - v - w - x)$
4. $(a - b + c - d - e + m) - (f + r + s - v - w - x)$
5. $\{a - (b - c)\} - \{d + (e - m)\} - \{f + (r + s)\} + \{v + (w + x)\}$
6. $\{(a - b) + c\} - \{(d + e) - m\} - \{(f + r) + s\} + \{(v + w) + x\}$
7. $\{a - (b - c + d)\} - \{e - (m - f - r)\} - \{s - (v + w + x)\}$
8. $\{(a - b + c) - d\} - \{(e - m + f) + r\} - \{(s - v - w) - x\}$
9. $\{a - (b - c) - d\} - \{e - (m - f) + r\} - \{s - (v + w) - x\}$
10. $\{a - b + c - (d + e - m)\} - \{f + r + s - (v + w + x)\}$
11. $\{(a - b + c - d) - e + m\} - \{(f + r + s - v) - w - x\}$
12. $\{a - (b - c) - d - (e - m)\} - \{f + (r + s) - v - (w + x)\}$

Exercise XI.

1. $3a - 3b$; $4ax + 4b^2x - 4x^3$; $3p^2x - 3bp^2x - 3c^2p^2x.$
2. $am - b^2m + m^2p + x^2 - 3ax^2 - bx^2 - 3m^2x^2 + bm^2x^2 + m^4x^3.$
3. $7 + ax + 3ay - 4bx + 4xy - ac^2 - 3c^2y - m^2y.$
4. $a^2m - a^2n - 2acp + 2acq - c^2m + c^2n.$
5. $a - b - \frac{x}{z^3} - \frac{y}{z^3} + \frac{c}{z^3} - \frac{d}{z^3} - \frac{m}{z^3}$

6. $m + \frac{a}{xyz} - \frac{b}{xyz} + \frac{c}{xyz} + \frac{d}{xyz}$.

7. $amx - axy - a^2c - abc + ay - \frac{6a}{2a-c} + \frac{m}{2a-c} - \frac{3p}{2a-c}$.

8. $3bcd - 3abd + 3bfm - 3bfn - \frac{2c}{5x^2} - \frac{3m}{5x^2} + \frac{4p}{5x^2}$.

Exercise XII.

1. $5am + (1 + 9a)x + (3 + 15a - 2m)y$.
2. $(4 + m)a + (2m + 3a)x + (3x - 4 + m + 3a)y$.
3. $5(2a - x - bc) + 2(b - 2c) - 3m$.
4. $(2a + m)x - (3am + 2c + a)xy + (3a - 2cm - b - f)y^2$.
5. $\{3(a + b + c) - (6m - e)a\}y - \{c + 2(1 + 3a)m\}x - c(2 - a)z$.
6. $\{11(a + b)m + 3(cy + a)\}y - \{3(a - b + c) + 2(a + 3)c\}xy + 3(m + a)c - 2acp$.

Exercise XIII.

1. $a^4 - 4a^3y + 7a^2y^2 - 6ay^3 + 2y^4$; $a^5 - a^4b - 2a^3b^2 + 2a^2b^3 + ab^4 - b^5$.
2. $2a^3m^3 + 10a^2m^2xy - 3amx^2y^2 - 9x^3y^3$; $9a^4x^4 - 3a^2x^3 - 3a^2x - 9a^3x^5 + 3ax^4 + 3ax^2$.
3. $a^5 + m^5$; $2a^4 - 2a^3xy - 2a^3x + 4a^2y^2 + 2a^2x^2y - 2axy^2 - 2axy^3 + 2y^4$.
4. $x^3 - 7x^2 + 5x + 28$ $a^8 - a^2$.
5. $a^5 - 4a^3b^2 + 4a^2b^3 - 17ab^4 - 12b^5$.
6. $a^2b^2 - a^2c^2 + 2abc^2 - b^2c^2$.
7. $a^6 - 6a^4b^2 - 10a^3b^3 - 6a^2b^4 + b^6$.
8. $3x^3 + 4abx^2 - 6a^2b^2x - 4a^3b^3$; $x^4 + x^3 - 4x^2 + 5x - 3$.
9. $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$.
10. $6y^6 - 5x^2y^5 - 6x^4y^4 + 21x^2y^3 + x^4y^2 + 15x^4$; $a^{m+n} + a^nb^m + a^mb^n + b^{m+n}$.
11. $30a^5 - 5a^4 - 207a^3 - 178a^2 + 78a + 72$.
12. $a^2x^2 + a(b + c)xy + bcy^2$; $a^{2m+1} - a^{m+1}b^n - a^mb^{n-p} + a^{m+1}c^p + b^{2n-p} - b^{n-p}c^p$.
13. $a^{m+2} - a^2c^p + a^2q^r - a^mm^3 + c^pm^3 - m^3q^r + a^mx^a - c^px^a + q^rx^a$
14. $a^5 - 2a^4x + 3a^3x^2 - 3a^2x^3 + 2ax^4 - x^5$.
15. $\{6ac(2c - m) - 3bc(2c - 12a + 3b - m) - 9b(2a - m)\}m + \{2am(c + 3b) + 4ac(c - 3b) + 2bc(c + 3b) - bm(c + 3b)\}x$.

Exercise XIV.

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1. $5bc$; $6x^2y$; $3a$; $-xyz^3$
2. $-2bcm$; $-ax^2$; $9mxy$; $3x^4$
3. $\dfrac{3b^2c}{5xy}$; $-\dfrac{17bx}{11m}$; $\dfrac{3axy}{5z^4}$; $-\dfrac{b^3}{16x^2}$

Exercise XV.

1. $\dfrac{3y^2}{c} - \dfrac{27bc}{4x} + \dfrac{3xy}{c} - \dfrac{2m}{x}$.
2. $\dfrac{3y}{5a} - \dfrac{11}{35xy} + \dfrac{2x}{5a} - \dfrac{7y}{5ax}$
3. $4a^2 + m - \dfrac{3a}{2} + \dfrac{5mxy}{2a^2}$.
4. $-\dfrac{abc}{4mxy} - \dfrac{a^2c^2}{3mxy} + \dfrac{4ay}{3m} + \dfrac{5a^2}{2xy}$

Exercise XVI.

1. $x - y$; $a^2 + 2ab + b^2$
2. $m^2 + 2mx + x^2$
3. $9x^4 - 10x^3 + 5x^3 - 30x$
4. $a^2 + 4ab + b^2$; $x^2y^2 + xy + 1$
5. $x^4 + 2x^3 + x^2 - 4x - 11$
6. $a^5 - a^4m - am^4 + m^5$
7. $1 - a + a^2 - a^3 +$ &c.; $a + a^2 + a^3 +$ &c.; $1 - 2m + 2m^2 - 2m^3$ + &c.; and $1 - 3x + 7x^2 - 10x^3 + 17x^4 -$ &c
8. $2a^2 - 6am + 4m^2$
9. $2a^3 - 3ab^2 + 5b^3$
10. $a + b + c$
11. $36x^3 - 27x^2y - 16xy^2 + 12y^3$
12. $2a^m - 3h$

Exercise XVII.

1. $a^2 - 6ay + 9y^2$; $9a^2 + 12ax + 4x^2$; $9x^2y^2 - 42xy + 49$; $4a^2x^4 - 12ax^3 + 9x^2$; $4a^2 + 12a^2xy^2 + 9a^2x^2y^4$

2. $a^2 - 9x^2$; $4a^2 - 9y^2$; $9a^2b^2 - x^2y^2$; $4m^4 - 9x^2y^6$.

3. $9a^2 - 4x^2y^2$; $4a^2 - 49$; $9 - x^2$; $4 + 20ay + 25a^2y^2$; $9a^2 - 24ax^2y^3 + 16x^4y^6$.

4. $x^2 + 5x - 66$; $9a^2 + 9a - 10$; $x^2 - 13x + 36$; $x^2 - 4x - 21$; $x^2 - 3x + 2$.

5. $a^6 + a^5x + a^4x^2 + a^3x^3 + a^2x^4 + ax^5 + x^6$; $a^5 - a^4x + a^3x^2 - a^2x^3 + ax^4 - x^5$; $m^4 - m^3a + m^2a^2 - ma^3 + a^4$; $c^4 + x^4$ is not div. by $c + x$. (See Theorem XIII.)

6. $a^{10} - a^9xy + a^8x^2y^2 - a^7x^3y^3 + a^6x^4y^4 - a^5x^5y^5 + a^4x^6y^6 - a^3x^7y^7 + a^2x^8y^8 - ax^9y^9 + x^{10}y^{10}$; $a^8m^8 + a^7m^7r + a^6m^6r^2 + a^5m^5r^3 + a^4m^4r^4 + a^3m^3r^5 + a^2m^2r^6 + amr^7 + r^8$; $a^8 + m^8s^8$ is not div. by $a - ms$ (see Theorem XI); $a^3 + a^2yz + ay^2z^2 + y^3z^3$.

7. $x + 4$; $x + 8$; $2x - 1$; $3a^3x - a^2$.

Exercise XVIII.

1. $a^2 - 2ab + b^2 - c^2$; $a^2 - b^2 + 2bc - c^2$; $a^2 - b^2 - 2bc - c^2$.

2. $16 - 9a^2 + 12ac - 4c^2$; $4a^2 - x^2 + 6m^2x - 9m^4$; $4x^2y^2 - 4a^2 + 12ay - 9y^2$.

3. $4a^2 - 12ac + 9c^2 - 4x^2 + 12xy - 9y^2$; $a^2 + 6ad + 9d^2 - 4c^2 - 16cm - 16m^2$.

4. $9a^2 - 6am^2 + m^4 - 4 + 4xy - x^2y^2$; $4a^4 - 12a^2x^2 + 9x^2 - 1 - 2y^2 - y^4$.

5. $37ab - 10a^2 - 26b^2 - 36$

6. $75a^2 - 12axy + 23x^2y^2$

7. $1 - x^{128}$

8. $a^{n-1} - x^{n-1}y^{n-1}$

Exercise XIX.

1. $(a - m)(a^2 + am + m^2)$

2. $(a + c)(a^4 - a^3c + a^2c^2 - ac^3 + c^4)$

3. Not resolvable.

4. $(a^3 + b^3)(a^3 - b^3)(a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2)$

5. $(a - x)(a^2 + ax + x^2)(a^6 + a^3x^3 + x^6)$

6. $(a - b)(a^{10} + a^9b + a^8b^2 + a^7b^3 + a^6b^4 + a\ b^5 + a^4b^6 + a^3b^7 + a^2b^8 + ab^9 + b^{10}$

7. $(a^2 + m^2x^2)(a + mx)(a - mx)$

8. $(2a+x)(16a^4-8a^3x+4a^2x^2-2ax^3+x^4)$

9. $(9+4c^2)(3+2c)(3-2c)$

10. $(3m-2c)(81m^4+54m^3c+36m^2c^2+24mc^3+16c^4)$

11. $(a+x)(a^6-a^5x+a^4x^2-a^3x^3+a^2x^4-ax^5+x^6)(a^{14}-a^7x^7+x^{14})$

12. $(a^4+m^4)(a^{16}-a^{12}m^4+a^8m^8-a^4m^{12}+m^{16})$

13. $(c^8+x^8)(c^{16}-c^8x^8+x^{16})$

14. $(x^2+m^2)(x^8-x^6m^2+x^4m^4-x^2m^6+m^8)(x^{20}-x^{10}m^{10}+m^{20})$

15. $(a-c)(a+c)(a^2+c^2)(a^4+c^4)(a^8+c^8)(a^2+ac+c^2)(a^2-ac+c^2)(a^4-a^2c^2+c^4)(a^8-a^4c^4+c^8)(a^{16}-a^8c^8+c^{16})$

16. $(a^{32}+m^{32})(a^{64}-a^{32}m^{32}+m^{64})$

17. $(a+c)(a-c)(a^2+c^2)(a^2-ac+c^2)(a^2+ac+c^2)(a^4-a^2c^2+c^4)(a^6-a^3c^3+c^6)(a^6+a^3c^3+c^6)(a^{12}-a^6c^6+c^{12})(a^{18}-a^9c^9+c^{18})(a^{18}+a^9c^9+c^{18})(a^{36}-a^{18}c^{18}+c^{36})$

18. $(m^{16}+c^{16})(m^{32}-m^{16}c^{16}+c^{32})(m^{96}-m^{48}c^{48}+c^{96})$

19. $(a^2+m^2)(a^{12}-a^{10}m^2+a^8m^4-a^6m^6+a^4m^8-a^2m^{10}+m^{12})$

20. $(am-p)(a^2m^2+amp+p^2)(a^6m^6+a^3m^3p^3+p^6)(a^{18}m^{18}+a^9m^9p^9+p^{18})(a^{54}m^{54}+a^{27}m^{27}p^{27}+p^{54})$

Exercise XX.

1. $a-2x$

2. $14a^2-43x^2-4ax$

3. $3\sqrt{3}+6\sqrt{6}+2\sqrt{5}-8\sqrt{x}-\sqrt{2}-4ax^2+a^2x^2-3a^2x$

4. $a^{c+m}+a^cx^{p+q}-a^mx^{m-p}-x^{m+q}$

5. $a^{n-1}-a^{n-2}x+a^{n-3}x^2-a^{n-4}x^3+a^{n-5}x^4-\dfrac{a^{n-5}x^5+x^n}{a+x}$

6. $(x-17)(x+3)$

7. $1+1+1+1+1+$ &c., to infinity, $=\infty$

8. $(a-x)(a+x)(a^2+ax+x^2)(a^2-ax+x^2)(a^6+a^3x^3+x^6)(a^6-a^3x^3+x^6)$

9. $x^2m^2(a^2x-2p)^2$

10. $-89\frac{1}{2}$

11. x^6-2x^3+1 and $a^5-4a^3b^2-8a^2b^3-17ab^4-12b^5$

12. x^2-ax+b

13. $(a^{32}+m^{32})(a^{16}+m^{16})(a^8+m^8)(a^4+m^4)(a^2+m^2)(a+m)(a-m)$

14. $a^{44}-c^{44}$ 15. $\frac{1}{8}$ 16. $2a(a^2+3b^2)$ 17. $2a(a-m)$

Exercise XXI.

1. $6ab^2m$	4. $x+2$	7. $x-7$
2. $3a^2m^2$	5. $a^2(a-x)$	8. $a^2(x-1)$
3. xy	6. $m^2(a^2-m^2)$	9. $x-1$

Exercise XXII.

1. $x+2$	5. $a-2b$	9. $a-2$
2. $x-2$	6. $a-b$	10. $4(a-b)^2$
3. $a-x$	7. $5x^2-3x+4$	11. a^3+a^2-5a+3
4. $x+4$	8. $ab-by$	12. $a^2+2ab-2b^2$

Exercise XXIII.

1. $12a^2b^2x^2y^2$	6. $36a^7-36a^6b-36ab^6+36b^7$
2. $12a^2x^2y^2z^2$	7. x^3-10x^2+21x
3. $(x^3-x^2y-xy+y^2)^2$	8. $a^4-a^3-ax^3+x^3$
4. $x^6+x^5y+x^4y^2-x^2y^4-xy^5-y^6$	9. $a^4-10a^3+35a^2-50a+24$
5. $4x^5-4x^4-4x^3+4x^2$	10. $60(a^{10}+a^9b-a^8b^2-2a^7b^3-2a^6b^4+2a^4b^6+2a^3b^7+a^2b^8-ab^9-b^{10})$

Exercise XXIV.

1. $\frac{a-b}{x-y}$	9. $\frac{a^2-ab+b^2}{a-b}$	17. $\frac{x+2y+3y^2}{2x^2-3xy-5y^2}$
2. $\frac{2a+m-m^2}{3a^2+m}$	10. $\frac{a-b}{a^2+ab+b^2}$	18. $\frac{a-b}{a^2+ab+b^2}$
3. $\frac{c}{n}$	11. $\frac{a^3+b^3}{a^3-b^3}$	19. $\frac{a^2+m^2}{a-m}$
4. $\frac{a^2b}{x}$	12. $\frac{a^4+a^2m^2+m^4}{1}$	20. $\frac{c+d}{m+2p}$
5. $\frac{ac^2}{a+c}$	13. $\frac{a^2+m^2}{a^3}$	21. $\frac{x+a}{x+c}$
6. $\frac{axy^3}{a^2xm+ay+x^2y^2z^3}$	14. $\frac{7}{11}$	22. $\frac{2x^2+3x-5}{7x-5}$
7. $\frac{3-5x}{x}$	15. $\frac{x-4}{x+3}$	23. $\frac{a+m}{x^2-a^2+2am-m^2}$
8. $\frac{1}{a+m}$	16. $\frac{2x+3}{x-4}$	24. $\frac{a^8-a^4x^4+x^8}{a^{16}-a^{12}x^4+a^8x^8-a^4x^{12}+x^{16}}$

Exercise XXV.

$\dfrac{-axy+3-2a}{ax}$

2. $\dfrac{\quad}{a-1}$

3. $\dfrac{3ax+9a-yx-3y-3a^2+30}{x+3}$

4. $\dfrac{3ax-3ay-2a-y^2}{x-y}$

5. $\dfrac{3a^2x-ay^2-2xy^2+am+mx}{a+x}$

6. $\dfrac{2xy(z+m)}{z+2m}$

7. $\dfrac{2b(3a^2+b^2)}{a+b}$

8. $\dfrac{2m^2}{a^2+m^2}$

9. $\dfrac{2ax}{a^2+x^2}$

Exercise XXVI.

1. $4m-4+\dfrac{1}{5m}$

2. $a+x+\dfrac{2x^2}{a-x}$

3. $x+y+x^2-xy+y^2-\dfrac{y^3(1+y)}{x+y}$

4. $5m^2+5mp+5p^2+\dfrac{3}{m-p}$

5. $a-\dfrac{1}{b}$

6. $1+5a-\dfrac{b(4a+1)}{m+b}$

Exercise XXVII.

1. $\dfrac{acdm}{bcdm};\ \dfrac{b^2dm}{bcdm};\ \dfrac{bc^2m}{bcdm};\ \dfrac{bcdx}{bcdm}$

2. $\dfrac{xy}{mxy},\ \dfrac{am}{mxy},\ \dfrac{by}{mxy}$

3. $\dfrac{8bxy}{12abxy},\ \dfrac{3a^2xy}{12abxy},\ \dfrac{6abm}{12abxy}$

4. $\dfrac{(1+m)^2}{1-m^2},\ \dfrac{(1-m)^2}{1-m^2}$

5. $\dfrac{x(x^2-y^2)}{x(x^2+y^2)},\ \dfrac{x+y}{x(x^2+y^2)}$

6. $\dfrac{6x^2+6xy}{2(x^2-y^2)},\ \dfrac{8x+2y}{2(x^2-y^2)},\ \dfrac{2x^2-5xy+3y^2}{2(x^2-y^2)}$

7. $\dfrac{18a^3m}{6a^2m(2+x)},\ \dfrac{16a^2-4a^2x^2}{6a^2m(2+x)},\ \dfrac{6m+3mx}{6a^2m(2+x)}$

8. $\frac{3ax^2 - 3a}{3(x^4 - 1)}$, $\frac{4x^3 - 4x}{3(x^4 - 1)}$, $\frac{3x^2 + 3}{3(x^4 - 1)}$ and $\frac{3x^3 + 2x^2 - 3x - 2}{3(x^4 - 1)}$

9. $\frac{6a^3 - 6a^2b}{6a^3(a^2 - b^2)}$, $\frac{2a}{6a^3(a^2 - b^2)}$, and $\frac{a - b}{6a^3(a^2 - b^2)}$

Exercise XXVIII.

1. $\frac{4am + 3m - 2bc}{2bm}$
2. $\frac{x^2y + 3xy + 2a - 2b}{xy^2 + 3y^2}$
3. $\frac{4ab}{b^2 - a^2}$
4. $\frac{332x + 63x^2}{63}$
5. $\frac{x^3 + xy^2 + y^3}{(x + y)^3}$
6. 0
7. $\frac{m^2 - 2mp - p^2}{m^2 - p^2}$
8. $\frac{14 - 12a}{1 - 4a^2}$
9. $\frac{1}{2 + x}$
10. $\frac{2x}{b}$
11. 0
12. $\frac{2ac - 2bc}{ab + bc + ac + b^2}$
13. $\frac{14x - 20x^3}{1 - 5x^2 + 4x^4}$
14. $\frac{m}{abc}$

Exercise XXIX.

1. $\frac{3x^2}{5a}$
2. 2
3. $\frac{2a - 2b}{3y}$
4. $\frac{3x^2 - 3}{2a + 2b}$
5. $\frac{a(a - b)}{x}$
6. $-\frac{a^3 + a^2m + am^2 + m^3}{my}$
7. $\frac{4ax - 4x^2}{3}$.
8. $\frac{x^2 - 11x + 28}{x^2}$
9. $\frac{am}{f^2y^{15}}$
10. $\frac{(a - 2)^2}{2a}$
11. $\frac{x + a}{x + d}$
12. $\frac{x^2 + 4x - 21}{x^2 - 19x + 88}$
13. $\frac{a^4 + a^2 + 1}{a^2}$
14. 1

Exercise XXX.

1. $\frac{y}{x^2}$
2. $\frac{a + x}{a - x}$
3. $\frac{a - b}{a + b}$
4. $3a^2y - 6a^2 + 3axy - 6ax$
5. $\frac{x - 3}{x - 7}$
6. 1

7. 1 8. $\dfrac{3a^3 - 3a}{x^2 - 1}$ 9. 1 10. $\dfrac{ab}{a^2 + b^2}$

Exercise XXXI.

1. $\dfrac{5a - 5b}{10a + 9b}$
2. $\dfrac{7a - 2x}{21}$
3. $\dfrac{ax}{a + 2x}$
4. $\dfrac{63 - 36x}{30x - 10}$
5. $\dfrac{45 - 18x + 18a}{20a + 20x - 12}$
6. $\dfrac{4a}{1 + 4a^2}$
7. $-a$
8. a
9. $-\dfrac{1}{x^2y^2}$
10. $\dfrac{df + c}{df - c}$
11. $\dfrac{1 + 4m^2}{4m^3 - m}$

Exercise XXXII.

1. $4\frac{8}{19}$
2. 5
3. 105
4. $2\frac{5}{16}$
5. 19
6. 7
7. $16\frac{2}{3}$
8. 41
9. 3
10. $17\frac{61}{151}$
11. 9
12. 4
13. 5
14. 12
15. 8
16. 9
17. 120
18. -10
19. 4
20. 15
21. 8
22. 80
23. 4
24. 0
25. 4
26. $\dfrac{c - b}{a}$
27. $\dfrac{3b(b + c)}{11a}$
28. $\dfrac{a - b^2}{8b - 3a - 6}$
29. $\dfrac{6a^2}{4a^3b + 2a - ab - b^2}$
30. $\dfrac{20ab + b^2c + 5ac - 15abc}{15b + abc - 10c}$
31. $\dfrac{bdf}{bd + ad + bc}$
32. $\dfrac{10a - 4ab^2}{3b + 4a}$
33. $\dfrac{bc(b - a)}{ab - a^2 - b^2}$
34. $\dfrac{b^2 + 19ab - 4a^2}{2a + 8b - 2}$
35. $\dfrac{a}{2(2b - 1)}$
36. $\dfrac{ab}{a + b}$
37. $4\frac{76}{231}$
38. $\dfrac{297a}{650 - 99a}$
39. $\frac{1}{4}$
40. $\dfrac{180 + 39b - 35c}{72a}$
41. $\dfrac{3ab - ac - a^2b^2}{a^2 + 3ab - b^2 - c - a}$

Exercise XXXIII.

1. 30 ; 17 5. 12 ; 18 ; 24 9. 56 13. 14
2. 21 ; 42 6. \$560 10. 14 14. 23
3. \$52·50 7. 30 11. 26 15. $38\frac{9}{13}$
4. 64 8. 163 12. 102 16. \$3 ; 12, 7
17. $26\frac{3}{5}$ miles 18. $134\frac{22}{27}$ hours 19. 1803 ; 1689
20. A = \$2542 ; B = \$2422 ; C = \$2436
21. Music \$0·55$\frac{5}{14}$; drawing \$0·32$\frac{1}{7}$
22. 70 vol. Science ; 210 vol. Travels ; 210 vol. Biography ; 315 vol. History ; 630 vol. General Literature.
23. Niagara river, $34\frac{1}{5}$ miles ; Rideau canal, $130\frac{1}{6}$ miles.
24. $2\frac{34}{37}$ days.
25. $\dfrac{n+a-c}{2}$ and $\dfrac{n-a+c}{2}$
26. (I) 1 h. $5\frac{5}{11}$ m. ; (II) 12 h. $32\frac{8}{11}$ m. ; (III) 12 h. $16\frac{4}{11}$ m.
27. \$155 and \$220
28. $19\frac{41}{45}$ days.
29. A, \$3594·50 ; B, \$1055·57$\frac{1}{7}$; C, \$1795·03 ; D, \$743·89$\frac{6}{7}$
30. $9\frac{81}{131}$ days. 31. 68
32. \$8142·85$\frac{5}{7}$ 33. 72 lbs.
34. \$11100 35. $\dfrac{abn}{b-a}$ feet.
36. 11 times, viz.: 1 h. $5\frac{5}{11}$ m.; 2 h. $10\frac{10}{11}$ m.; 3 h. $16\frac{4}{11}$ m.; &c.
37. $90\frac{3}{10}$ and $5\frac{7}{10}$
38. A's = \$808·42$\frac{2}{19}$; B's = \$538·94$\frac{14}{19}$; C's = \$1212.63$\frac{3}{19}$
39. 820 miles ; 15 m. per h. down ; 10 m. and 12 m. per h. up.
40. 5 ; \$9000 41. 18
42. A's = \$657·14$\frac{2}{7}$; B's \$731·42$\frac{6}{7}$; C's = \$711·42$\frac{6}{7}$
43. 2575 44. $\dfrac{na}{m+n}$ and $\dfrac{ma}{m+n}$ 45. 15 and 45
46. 36 weeks. 47. $\dfrac{a}{1+m+n}$; $\dfrac{na}{1+m+n}$; and $\dfrac{ma}{1+m+n}$
48. $\dfrac{anq}{nq+mq+np}$; $\dfrac{amq}{nq+mq+np}$ and $\dfrac{anp}{nq+mq+np}$ 49. 189

Exercise XXXIV.

1. $x = 2$; $y = 3$ 2. $x = 5$; $y = 6$ 3. $x = 20\frac{1}{2}$; $y = 5\frac{1}{2}$
4. $x = 4$; $y = 10$ 5. $x = 7$; $y = 3$ 6. $x = 24$; $y = 30$

7. $x = 2\frac{144}{149}$; $y = 3\frac{35}{149}$ 8. $x = 12$; $y = 0$ 9. $x = 3$; $y = 5$

10. $x = \dfrac{2a + 3b}{19}$; $y = \dfrac{5a - 2b}{19}$ 11. $x = \dfrac{an - bm}{4a - 3b}$; $y = \dfrac{4m - 3n}{4a - 3b}$

12. $x = \dfrac{2ac - b^2}{3ab}$; $y = \dfrac{ac - 2b^2}{3ab}$ 13. $x = \dfrac{a^2 + b}{2a}$; $y = \dfrac{b - a^2}{2a}$

14. $x = \dfrac{amc(a + c + m)}{mc + ma - ac + c^2}$; $y = \dfrac{acm(2c - m)}{cm + am - ac + c^2}$

15. $x = \dfrac{mq + bn}{aq + bn}$; $y = \dfrac{bn + mq}{ab - bm}$

16. $x = 8$; $y = 3$

17. $x = 8$; $y = 9$

18. $x = \dfrac{a(c^2p - a^2 - c^2)}{c^2 - a^2}$; $y = \dfrac{c(a^2p - c^2 - a^2)}{c^2 - a^2}$

19. $x = 9$; $y = 7$

20. $x = \dfrac{ab}{a - b}$; $y = \dfrac{ab}{a + b}$

Exercise XXXV.

1. $x = 11$; $y = 2$; $z = 3$ 2. $x = 2$; $y = 0$; $z = 3$

3. $x = 1$; $y = 2$; $z = -3$ 4. $x = 4$; $y = 1$; $z = -2$

5. $x = 1\frac{3}{4}$; $y = -2$; $z = 2$; $v = -1\frac{3}{4}$ 6. $x = 2$; $y = 3$; $z = 4$

7. $x = 1\frac{1}{3}$; $y = 4$; $z = \frac{4}{5}$

8. $x = \dfrac{5m + 16n - 3b}{76}$; $y = \dfrac{11b + 7m - 8n}{76}$; $z = \dfrac{23b + 4n - 13m}{76}$

9. $x = \dfrac{c^3 - b^2c + a^2b}{ab^2 + ac^2}$; $y = \dfrac{2bc - a^2}{b^2 + c^2}$; $z = \dfrac{b^3 - bc^2 + a^2c}{ab^2 + ac^2}$

10. $v = 2$; $x = 5$; $y = 6$; $z = 10$

11. $x = b + c - a$; $y = a + c - b$; $z = a + b - c$

12. $x = \dfrac{ap - am + an - m}{2a^2 - a - 1}$; $y = \dfrac{am - n + ap - an}{2a^2 - a - 1}$; $z = \dfrac{am - ap + an - p}{2a^2 - a - 1}$

Exercise XXXVI.

1. 4 and 2

2. $\dfrac{ac}{b + c}$ and $\dfrac{ab}{b + c}$

3. \$15 and \$0·40
4. $125\frac{5}{11}$ yds. long and $40\frac{4}{11}$ yds. wide.
5. 12 and 15
6. 84 and 60
7. 32 and 16
8. -7; $-\frac{1}{5}$ and $-5\frac{2}{3}$
9. 380 sulphur; 620 charcoal; and 3000 saltpetre.
10. 16; 24; and 32
11. $40\frac{8}{36}$ shillings, or $44\frac{1}{2}$ ten cent pieces.
12. 29 lines and 32 letters. 13. 78
14. 116 ten and 280 twenty-five cent pieces.
15. $\dfrac{c}{(a-1)(b-d)}$.
16. 5 inside and 9 outside passengers; \$4½ and \$2½
17. 36 18. 432
19. $\dfrac{(c-a)p}{c-b}$ and $\dfrac{(a-b)p}{c-b}$
20. \$81, \$41, \$11, \$21, \$11 and \$6

Exercise XXXVII.

1. $8a^6$; $9a^2b^6$; $16m^4$; $3ab^2c^3$; 1; 1; $3a^2xy^3$
2. a^{12}; $-128a^{14}b^7c^{14}$; $-\frac{1}{8}a^3b^3c^9$; $\frac{1}{9}x^2y^6$; $-32m^5x^{10}y^{16}$
3. 1; $a^8x^{16}y^{24}z^{32}$; $27a^3y^9$; $-27a^3y^9$; $81a^4y^{12}$; $81a^4y^{12}$

Exercise XXXVIII.

1. $a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^6 - 36a^2b^7 + 9ab^8 - b^9$
2. $c^4 + 4c^3x + 6c^2x^2 + 4cx^3 + x^4$
3. $x^{10} - 10x^9y + 45x^8y^2 - 120x^7y^3 + 210x^6y^4 - 252x^5y^5 + 210x^4y^6 - 120x^3y^7 + 45x^2y^8 - 10xy^9 + y^{10}$
4. $a^{11} + 11a^{10}m + 55a^9m^2 + 165a^8m^3 + 330a^7m^4 + 462a^6m^5 + 462a^5m^6 + 330a^4m^7 + 165a^3m^8 + 55a^2m^9 + 11am^{10} + m^{11}$
5. $16 - 32a + 24a^2 - 8a^3 + a^4$
6. $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$
7. $64a^6 + 576a^5 + 2160a^4 + 4320a^3 + 4860a^2 + 2916a + 729$
8. $243 - 810m + 1080m^2 - 720m^3 + 240m^4 - 32m^5$

9. $243a^5 - 810a^4y + 1080a^3y^2 - 720a^2y^3 + 240ay^4 - 32y^5$

10. $8b^3 - 60b^2c + 150bc^2 - 125c^3$

11. $81x^4 - 432x^3y + 864x^2y^2 - 768xy^3 + 256y^4$

12. $a^5b^5 + 15a^4b^4c + 90a^3b^3c^2 + 270a^2b^2c^3 + 405abc^4 + 243c^5$

13. $8a^3c^3 - 12a^2c^2xyz + 6acx^2y^2z^2 - x^3y^3z^3$

14. $a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$

15. $16a^4 - 32a^3b - 32a^3c + 24a^2b^2 + 48a^2bc + 24a^2c^2 - 8ab^3 - 24ab^2c - 24abc^2 - 8ac^3 + b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4$

16. $32a^5 + 160a^4b + 320a^3b^2 + 320a^2b^3 + 160ab^4 + 32b^5 + 240a^4c - 960a^3bc - 1440a^2b^2c - 960ab^3c - 240b^4c + 720a^3c^2 + 2160a^2bc^2 + 2160ab^2c^2 + 720b^3c^2 - 1080a^2c^3 - 2160abc^3 - 1080b^2c^3 + 810ac^4 + 810bc^4 - 243c^5$

17. $1 + 4x + 2x^2 - 8x^3 - 5x^4 + 8x^5 + 2x^6 - 4x^7 + x^8$

18. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5 + 10a^4c - 40a^3bc + 60a^2b^2c - 40ab^3c + 10b^4c + 40a^3c^2 - 120a^2bc^2 + 120ab^2c^2 - 40b^3c^2 + 80a^2c^3 - 160abc^3 + 80b^2c^3 + 80ac^4 - 80bc^4 + 32c^5$

Exercise XXXIX.

1. $4 + 2x - 11\frac{3}{4}x^2 - 3x^3 + 9x^4$

2. $x^2 + 2x^3 - x^4 - 2x^5 + x^6$

3. $4x^2 - 12x^3 + 7x^4 + 3x^5 + \frac{1}{4}x^6$

4. $1 - a + 4\frac{1}{4}a^2 - 4a^3 + 5a^4 - 4a^5 + a^6$

5. $1 + 2x - 2x^3 + \frac{5}{4}x^4 + \frac{5}{2}x^5 - \frac{3}{4}x^6 - x^7 + x^8$

6. $4a^2 - 4a^2x + 9a^2x^2 - 4a^2x^3 + 4a^2x^4$

7. $1 + 2bx + (b^2 - 2c)x^2 - 2bcx^3 + c^2x^4$

8. $a^2 - 2abx - (2ac - b^2)x^2 + (2ad + 2bc)x^3 - (2bd - c^2)x^4 - 2cdx^5 + d^2x^6$

9. $1 - 2a + a^2 + 2b^2x^2(1 - a) - 2c^3x^3(1 - a) + (2d^4 - 2ad^4 + b^4)x^4 - 2b^2c^3x^5 + (2b^2d^4 + c^6)x^6 - 2c^3d^4x^7 + d^8x^8$

10. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

11. $a^8 - 8a^7c + 28a^6c^2 - 56a^5c^3 + 70a^4c^4 - 56a^3c^5 + 28a^2c^6 - 8ac^7 + c^8$

12. $a^4x^4 - 8a^3x^3 + 24a^2x^2 - 32ax + 16$

13. $4 - 12x + 25x^2 - 26x^3 + \frac{61}{3}x^4 - 6x^5 + \frac{35}{12}x^6 - \frac{1}{3}x^7 + \frac{1}{9}x^8$

14. $1 - 4x + 2x^2 + 8x^3 - 9x^4 + 6x^6 - 4x^7 + x^8$

Exercise XL.

1. $\pm a^2$; $\pm xy$; $\pm 2am^2$; $\pm 8a$; $\pm 11a^3y^4$
2. $-3a$; $4a^2y^3$; $5ax^5$; $-2a^2y^4z$.
3. $\pm \frac{4a}{5b^2}$; $\bullet$; $\pm \frac{12x^2y^9}{9a^2b}$; $\pm \frac{8a^4}{25mx}$
4. $\frac{4a^4y^2}{3m}$; $\frac{2a^8x^6y^4}{6bc^2}$; $-\frac{7ab^3}{4m^2y^7}$
5. $\pm \frac{2a}{b^3}$; $\frac{2a^2x^4}{3y}$; $\pm \frac{3m^2x^2}{2a^2}$; $-\frac{a^2m^3}{x^4y^6}$

Exercise XLI.

1. $2a+3b$; $a-2x$; $2ax-7c$
2. $3am+5xy$; $4ax^2-b^2c^3$
3. $2x^2+3x-1$
4. x^2-y^2-1
5. $a+b-c$
6. $3a^2+2a+5$
7. $a+b+c+d$
8. $x^3-3x^2y+3xy^2-y^3$
9. $a^2-4ac+4c^2$
10. $1-y+3y^2+2y^3$
11. $2a^2+3ax+x^2$
12. x^2+y^2
13. $a^2-b^2+c^2-d^2$
14. $1-\frac{1}{2}x+x^2-\frac{1}{2}x^3$
15. $\frac{1}{2}x^2+\frac{x}{y}-\frac{y}{x}$

Exercise XLII.

1. $2x+3$
2. a^2+2a-4
3. $1-2a$
4. a^2-2a+1
5. $2ax-7bx^2$
6. $2x^2-3ax+4a^2$
7. x^2-x+1
8. $a+b+c+d+e$

Exercise XLIII.

1. $a^{\frac{1}{2}}$; $a^{\frac{2}{3}}$; $a^{\frac{5}{4}}$; $a^{\frac{1}{2}}b^{\frac{3}{2}}c$; $a^{\frac{4}{3}}b^{\frac{4}{3}}c^{\frac{4}{3}}$; $a^{\frac{6}{5}}b^{\frac{3}{5}}c^6$; $a^{\frac{mt}{n}}b^{\frac{rt}{n}}c^{\frac{st}{n}}$

2. $\sqrt[3]{a}$; $\sqrt[5]{b^2}$; $\sqrt[8]{c^3}$; $\sqrt{ab^3}$; $\sqrt[5]{abc}$; $\sqrt[7]{a^3b^3}$; $\sqrt[4]{(a^5b^3c)^3}$; $\sqrt[6]{(a^2b^4c^6m^7)^2}$; $\sqrt[my]{\{\sqrt[r]{a}\,\sqrt[ny]{b^2}\,\sqrt[my]{c^{rs}})\}^r}$

3. $2ab^{-1}m^{-1}$; $2a^{-1}$; $3am^{-1}$; $m^2a^{-1}c^{-2}$; $\frac{3}{4}abm^{-1}c^{-3}$; $\frac{2}{5}a^{\frac{1}{3}}c^{-1}$; $3a^{-\frac{7}{4}}c^{\frac{1}{2}}m^{\frac{7}{6}}$; $a^{-\frac{1}{3}}b^{-\frac{2}{3}}c^{-\frac{1}{3}}m^{-\frac{1}{12}}$ or $\left(ab^2cm^{\frac{1}{4}}\right)^{-\frac{1}{3}}$; $a^{\frac{1}{2}}m^{-\frac{1}{2}}$ or $(am^{-1})^{\frac{1}{2}}$

* See Art. 142.

4. $\frac{2}{a^{1}}$; $\frac{1}{cb^{-2}}$; $\frac{3}{a^{-1}m^{-1}c^{\frac{1}{2}}}$; $\frac{2}{3a^{-1}x^{2}y}$; $\frac{1}{a^{-1}b^{-2}c^{-\frac{1}{3}}}$; $\frac{3}{2a^{2}m^{\frac{1}{2}}x^{-1}y^{2}}$;
$\frac{4}{3a^{-1}c^{1}mx}$; $\frac{5}{2(ab)^{\frac{1}{2}}(mn^{2})^{\frac{1}{3}}x^{\frac{1}{6}}}$

5. $\frac{1}{a}$; $\frac{2a^{2}}{b^{3}}$; $\frac{3(acm)^{\frac{2}{3}}}{b^{3}}$; $\left(\frac{m}{b}\right)^{3}$; $\frac{3c^{3}m^{\frac{1}{2}}}{2a^{2}b^{3}}$; $ab^{\frac{1}{2}}c^{\frac{1}{3}}m^{\frac{3}{4}}$; $\left(\frac{b}{a}\right)^{\frac{1}{4}}$; $\frac{a^{3}}{b^{6}}$;
$\frac{c^{\frac{1}{3}}}{a^{\frac{2}{3}}b^{\frac{1}{6}}}$; $\left(\frac{b^{2}}{a}\right)^{mn}$

6. $\sqrt[6]{a^{19}}$; $a^{-\frac{49}{36}}$; $\frac{1}{\sqrt{a}}$

7. $\frac{c^{3}}{a^{2}}$; $a^{-\frac{3}{4}}$

8. $a^{17}b^{26}c^{8}$

9. $x^{mtr(1-q)+s(nt+pr)}$, $y^{4rs(q-1)-q(4n+7s)}$;

10. $a^{2}-4a^{\frac{3}{2}}b^{\frac{1}{2}}+6ab-4a^{\frac{1}{2}}b^{\frac{3}{2}}+b^{2}$

11. $a^{\frac{4}{3}}+a^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}$

12. $8x^{\frac{3}{2}}-4x^{\frac{1}{2}}y^{-1}+6x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{\frac{1}{3}}+2y^{-1}z^{\frac{1}{3}}-y^{-\frac{3}{2}}-2x^{\frac{1}{2}}z^{\frac{2}{3}}-y^{-\frac{1}{2}}z^{\frac{2}{3}}$

13. $2x^{-4}y^{-1}-3x^{-5}$

14. $a^{\frac{3}{8}}-a^{\frac{1}{4}}b^{-\frac{1}{8}}+a^{\frac{1}{8}}b^{-\frac{1}{4}}-b^{-\frac{3}{8}}$

15. $x^{-\frac{2}{3}}-x^{-\frac{1}{3}}+1-x^{\frac{1}{3}}+x^{\frac{2}{3}}$

16. $a^{3}-2a^{\frac{5}{2}}+3a^{2}-3a+2a^{\frac{1}{2}}+3-6a^{-\frac{1}{2}}+a^{-1}+4a^{-\frac{3}{2}}-a^{-2}-2a^{-\frac{5}{2}}+a^{-3}$

17. $a^{\frac{1}{3}}+1-a^{-\frac{1}{3}}$

18. $x^{\frac{2}{3}}-2x^{\frac{1}{3}}+3-2x^{-\frac{1}{3}}+x^{-\frac{2}{3}}$

19. $x^{-\frac{1}{3}}y-x^{\frac{1}{3}}y^{-1}$

20. $x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{6}}+3y^{\frac{1}{3}}$

Exercise XLIV.

1. $4^{\frac{1}{3}}$; $343^{\frac{1}{2}}$; $16^{\frac{1}{5}}$; $\left(\frac{9}{4}\right)^{\frac{1}{3}}$; $\left(\frac{16}{169}\right)^{\frac{1}{6}}$; $9^{\frac{1}{3}}$; $\left(\frac{1}{a^{5}}\right)^{\frac{1}{2}}$

2. $(a^2)^{\frac{1}{2}}$; $9^{\frac{1}{2}}$; $\left(\frac{81}{4}\right)^{\frac{1}{2}}$; $(4a^2)^{\frac{1}{2}}$; $(9a^4b^2)^{\frac{1}{2}}$; $(16x^4y^6)^{\frac{1}{2}}$; $\left(\frac{1}{a^3}\right)^{-\frac{1}{3}}$

$\left(\frac{1}{27}\right)^{-\frac{1}{3}}$; $\left(\frac{8}{729}\right)^{-\frac{1}{3}}$; $\left(\frac{1}{8a^3}\right)^{-\frac{1}{3}}$; $\left(\frac{1}{27a^6b^3}\right)^{-\frac{1}{3}}$; $\left(\frac{1}{64a^6y^9}\right)^{-\frac{1}{3}}$;

$(a^4)^{\frac{1}{4}}$; $(81)^{\frac{1}{4}}$; $\left(\frac{6561}{16}\right)^{\frac{1}{4}}$; $(16a^4)^{\frac{1}{4}}$; $(81a^8b^4)^{\frac{1}{4}}$ and $(256x^8y^{12})^{\frac{1}{4}}$

3. $\left(\frac{1}{a^4}\right)^{-\frac{1}{2}}$; $\left(\frac{1}{3}\right)^{-\frac{1}{2}}$; $\left(\frac{1}{4a^4b^6}\right)^{-\frac{1}{2}}$; $\left(\frac{1}{a^2c^4}\right)^{-\frac{1}{2}}$; $\left(\frac{25}{484}\right)^{-\frac{1}{2}}$;

$(81)^{-\frac{1}{2}}$; $\left(\frac{117649}{4096}\right)^{-\frac{1}{2}}$; $\left(\frac{z^6}{x^2y^4}\right)^{-\frac{1}{2}}$; $(a^6)^{\frac{1}{3}}$; $(\sqrt{27})^{\frac{1}{3}}$; $(8a^6b^9)^{\frac{1}{3}}$;

$(a^3c^6)^{\frac{1}{3}}$; $\left(\frac{10648}{125}\right)^{\frac{1}{3}}$; $\left(\frac{1}{729}\right)^{\frac{1}{3}}$; $\left(\frac{262144}{40353607}\right)^{\frac{1}{3}}$; $\left(\frac{x^3y^6}{z^9}\right)^{\frac{1}{3}}$

4. $\sqrt{48}$; $\sqrt{125}$; $\sqrt{124}$; $\sqrt{16a}$, $\left(\frac{3}{8}\right)^{\frac{1}{2}}$; $\left(\frac{b^2}{8a^4}\right)^{\frac{1}{3}}$

5. $\frac{2}{9}\sqrt{3ab}$; $\frac{a}{2b}\sqrt[3]{6}$; $\frac{1}{3}\sqrt{14}$; $\frac{4}{25}\sqrt[3]{20}$; $\frac{3a}{4b}\sqrt[3]{4b}$

6. $\sqrt[3]{108}$; $\sqrt[3]{8a}$; $\sqrt[3]{18}$; $\sqrt[4]{a^4c}$; $\left(\frac{200}{9a}\right)^{\frac{1}{3}}$; $\left(\frac{18m^2}{3125}\right)^{\frac{1}{5}}$; $(a^2m^2-p^2q^2)^{\frac{1}{2}}$

7. $3\sqrt[3]{5}$; $9\sqrt{2}$; $2\sqrt[4]{5}$; $21\sqrt[3]{12}$; $\frac{1}{14}\sqrt{21}$; $\frac{1}{a}(a^5m^5)^{\frac{1}{6}}$

$am(m^3-a^3+a^3m^3)^{\frac{1}{3}}$

8. $\frac{b}{6(a+x)}\sqrt{\{6a(a+x)\}}$; $\frac{cm}{bn}\sqrt{n}$; $a\sqrt[m]{(a^nx)}$

$\frac{z^2(a-z)^2}{c+z}\sqrt[q]{\{(b+z)(c+z)^{q-1}\}}$

9. $3\sqrt[3]{3}$ is the greater; $2\sqrt{11}$ is the greatest, and $3\sqrt[3]{2\frac{1}{5}}$ the least.

10. $50\sqrt{2}$; $4\sqrt{3}+2\sqrt{15}$.

11. $8\sqrt{7}-\sqrt[3]{3}$; $(3ab^2+2a^2-\frac{c^4}{b})\sqrt{ac}$

12. $(2a^pb^n+3a^{3-n}b-c^2)\sqrt[m]{a^3b^5}$

13. $15\sqrt{42}$; $60\sqrt{2}$; $70\sqrt{15}$; $24\sqrt[6]{12150}$

14. $4\sqrt[6]{32}$; $28a\sqrt[6]{a}$; $2\sqrt[6]{1944}$; $7\sqrt[6]{15}$

15. $xy\,(a^6b^4c^3x^6y^4z^3)^{\frac{1}{12}}$; $x^{\frac{3}{2}} + y^{\frac{3}{2}}$

16. $4\sqrt{6} + 6\sqrt{14} - 16\sqrt{15} - 12\sqrt{35}$; $3\sqrt{30} + \sqrt{6} - 24 - \frac{8}{5}\sqrt{5}$

17. $\frac{1}{4}\sqrt{6}$; $\frac{5}{12}\sqrt{14}$; $\frac{2}{3}\sqrt{10}$; $\frac{2}{39}\sqrt{130}$

18. $\frac{4}{7}\sqrt[6]{64827}$; $\frac{3}{10}\sqrt[6]{2000}$; $\frac{4}{9}\sqrt[6]{96}$; $\frac{4}{3ax}\sqrt{a^5x^5}$

19. 10; $2\sqrt[4]{3} - \frac{5}{6}\sqrt[12]{5038848} + \sqrt[12]{964467}$; $\frac{1}{a}\sqrt[5]{(a^3b^{n-3}\,c^{n+1}d)}$

20. -29; 1; -42; $2\frac{1}{450}$; $-\frac{7}{1400}$

21. $\frac{1}{17}(4\sqrt{5} - 2\sqrt{3})$; $\dfrac{2\sqrt{10} + 6\sqrt{3} + 2\sqrt{15} + 9\sqrt{2}}{-34}$;

$\dfrac{14\sqrt{6} + 8\sqrt{21} + 7\sqrt{22} + 4\sqrt{77}}{-28}$

22. $\dfrac{3\sqrt{3} + 3\sqrt{x}}{3 - x}$; $\dfrac{a - 2\sqrt{am} + m}{a - m}$; $\dfrac{30\sqrt{2} + 24\sqrt{15} + 30\sqrt{3} + 36\sqrt{10}}{-19}$

23. $\dfrac{x^2 - \sqrt{x^4 - x^2 - 2x - 1}}{x + 1}$

24. $\dfrac{2\sqrt{3} + \sqrt{30} - 3\sqrt{2}}{12}$; $\dfrac{26\sqrt{3} - 27\sqrt{6} + 51\sqrt{5} - 136}{92}$;

$\dfrac{136 - 3\sqrt{3} + 25\sqrt{6} - 14\sqrt{2}}{73}$

Exercise XLV.

1. $1 + \sqrt{5}$
2. $\sqrt{7} - \sqrt{5}$
3. $\frac{1}{2}\sqrt{126} + \frac{1}{2}\sqrt[3]{2}$
4. $\sqrt{22} - 1$
5. $\sqrt{6} - 2$
6. $2\sqrt{7} + \sqrt{14}$
7. $\frac{1}{2}\left(\sqrt{6} + \sqrt{2}\right)$
8. $5 - 3\sqrt{2}$
9. $\sqrt{a-1} - 1$
10. $\sqrt{a+b} + \sqrt{a-b}$
11. $\frac{1}{2}\left(\sqrt{26} + \sqrt{3}\right)$
12. $\frac{1}{2}(b^2 + \sqrt{a^2 - b^2})$

Exercise XLVI.

1. $\sqrt[4]{18} - \sqrt[4]{2}$
2. $\sqrt[4]{20} + \sqrt[4]{5}$
3. $\sqrt[4]{24} + \sqrt[4]{6}$
4. $\sqrt[4]{8} - \sqrt[4]{2}$

Exercise XLVII.

1. $8\sqrt{-3}$; $2a+(\sqrt{b}+\sqrt{c})\sqrt{-1}$
2. $(\sqrt{5}+\sqrt{7}+\sqrt{11})\sqrt{-1}$
3. $3+\sqrt{-2}$.
4. 50
5. $-29-6\sqrt{6}$
6. $\frac{1}{7}(\sqrt{2}-\sqrt{-5})$
7. $-a^{123}\sqrt{-1}$; $+1$; $\sqrt{-1}$; -1
8. $a^2-2a\sqrt{-a}-a$
9. $-4\sqrt{-1}-10\sqrt{2}$
10. $\sqrt{3}-\sqrt{-5}$
11. $\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{-2}$; $\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{-2}$
12. $7+3\sqrt{-2}$
13. $1+\sqrt{-2}$
14. $2-\sqrt{-3}$
15. a^2+b^2

Exercise XLVIII.

1. 4
2. 6
3. 49
4. $\dfrac{\sqrt{a}}{2+\sqrt{a}}$
5. 21
6. $\dfrac{a}{(\sqrt{a}-1)^2}$
7. $\pm\frac{1}{2}\sqrt{-3}$
8. 4
9. $\frac{2}{3}$
10. $\dfrac{4a}{a^2+4}$
11. $\dfrac{b^2-4a}{4a}$
12. $\dfrac{a^2-2ab}{3a-2b}$
13. $\dfrac{a^2b^2}{(a-b)^2}$
14. $\dfrac{(a-1)^2}{2a-1}$
15. 81
16. $\dfrac{a(b+c)}{b-c}$
17. $\frac{25}{16}$
18. $\left(\dfrac{c^2+b-a}{2c}\right)^2-b$
19. $2a$
20. $\dfrac{a(m^2+1)}{2m}$

Exercise XLIX.

1. ± 3
2. $\pm\frac{1}{6}$
3. ± 3
4. $\pm\frac{3}{2}$
5. ± 2
6. $\pm 5\sqrt{-1}$
7. ± 2
8. ± 6
9. $\pm 3\sqrt{a^2+1}$
10. $\pm\sqrt{2ab-b^2}$
11. $\pm\dfrac{2a}{5b}\sqrt{5}$
12. $\pm\left(\dfrac{d-b-1}{3a-c}\right)^{\frac{1}{2}}$
13. $\pm\left(\dfrac{a^2-1}{3+a^2}\right)^{\frac{1}{2}}$
14. $\pm\dfrac{(c-1)b}{\sqrt{2c-1}}$
15. $\pm 9\sqrt{2}$
16. $\pm\sqrt{a^2-\left(\dfrac{b^3-2a}{3b}\right)^3}$

Exercise L.

1. 5 or -9
2. 9 or -1
3. 10 or -2
4. 3 or -15
5. 5 or $-5\frac{2}{3}$
6. 3 or $1\frac{2}{3}$
7. 47 or $\frac{1}{5}$
8. 10 or -8
9. $1 \pm \sqrt{1-a^2}$
10. 7 or $-7\frac{4}{5}$
11. 4 or $-1\frac{1}{7}$
12. 4 or 3
13. 3 or $\frac{2}{3}$
14. $\frac{d}{c}$ or $-\frac{b}{a}$
15. 4 or 1 or $\frac{1}{2}(-3 \pm \sqrt{-7})$ or 0.
16. 15 or -14
17. 1 or -12
18. 0 or $\pm 2\sqrt{15} - 8$
19. 1 or $-\frac{3}{7}$
20. $\pm\sqrt{\frac{b+c}{f-a} + \left(\frac{b+c}{2f-2a}\right)^2} - \frac{b+c}{2f-2a}$
21. m or $-a$
22. $\frac{a+b \pm \sqrt{2(a^2+b^2)}}{\sqrt{ab}}$

Exercise LI.

1. 5 or $-5\frac{2}{3}$
2. 15 or -14
3. 5 or $-4\frac{1}{4}$
4. 25 or 1
5. 7 or $-7\frac{4}{5}$
6. $1\frac{1}{2}$ or $-3\frac{1}{2}$
7. 1 or $-\frac{4}{11}$
8. $\frac{c}{a}$ or $-\frac{b}{a}$
9. $\frac{1}{6}(4 \pm \sqrt{61})$
10. $\frac{1}{2-\sqrt{3}}$ or $\frac{2}{\sqrt{3}-2}$
11. $\pm\sqrt{9a^2+b^2} - 3a$
12. 3 or $-8\frac{7}{10}$
13. $\frac{1}{2}(5 \pm \sqrt{25-4m^2})$
14. $\frac{\sqrt{mn}}{\sqrt{m}-\sqrt{n}}$ or $\frac{\sqrt{mn}}{\sqrt{m}+\sqrt{n}}$
15. $\frac{1}{2}(a \pm \sqrt{a^2-4})$
16. 2 or $-2\frac{7}{13}$

Exercise LII.

1. $x^2 + 9x + 14 = 0$
2. $x^4 - 3x^3 - 6x^2 + 8x = 0$
3. $x^5 - 13x^3 + 36x = 0$
4. $x^6 - 6x^5 - 22x^4 + 174x^3 - 103x^2 - 600x + 700 = 0$
5. $x^6 - 20x^5 + 154x^4 - 590x^3 + 1189x^2 - 1190x + 456 = 0$
6. $x^6 - 14x^5 + 76x^4 - 206x^3 + 283x^2 - 140x = 0$.
7. $\frac{1}{2}(3 \pm \sqrt{-15})$
8. 3 or -1
9. $-10 \pm 6\sqrt{-5}$
10. 0 or $2 \pm \sqrt{-1}$
11. 0 or 5 or -2
12. 0 or 2 or -1

13. $c = 2$.

14. $cx^2 + bx + a = 0$

15. $p^2 - 2q$; $p^2 - 4q$; $\mp p\,(\sqrt{p^2 - 4q})$; $-\frac{p}{q}$; $(p^2 - q)\sqrt{p^2 - 4q}$

Exercise LIII.

1. 64 or 4
2. 81 or 1
3. ± 2 or $\pm\sqrt{10}$
4. 9 or $\sqrt[3]{1681}$
5. 10 or -2
6. ± 4 or $\pm\frac{1}{2}\sqrt{-62}$
7. 3 or $\sqrt[3]{-19}$
8. 4 or -1
9. 2 or -3
10. 4 or $7\frac{1}{9}$
11. 60 or 235
12. 4 or 1
13. 1, 0 or $\pm\sqrt{-1}$
14. 5 or $-2\frac{2}{15}$
15. 3 or $\pm\sqrt{-3}$
16. 3 or 2
17. 1, 1 or -2
18. $\pm\frac{c}{2}\sqrt{\frac{c^2 - 4b^2}{c^2 - a^2}}$
19. $\frac{1}{2}(b \pm \sqrt{b^2 - 2ab})$
20. 4 or -5
21. $\pm\sqrt{-1}$; -1; $\pm\sqrt{\frac{1}{2}(1 \pm \sqrt{-3})}$; 1; $\frac{1}{2}(1 \pm \sqrt{-3})$; $\frac{1}{2}(-1 \pm \sqrt{-3})$
22. 3, 2, or 1
23. 0 or $2 \pm \sqrt{3}$
24. 4, 5 or -1
25. $\frac{c^3 - ab}{a + b - 2c}$
26. $1\frac{2}{3}$, 1, or 2
27. $(\sqrt[3]{b} - \sqrt[3]{a})^3$
28. 2, $\frac{1}{4}$, or $\frac{1}{8}(9 \pm \sqrt{-31})$
29. 4, 9, or $\frac{1}{2}(-33 \mp \sqrt{-67})$
30. $1 \pm \sqrt{\pm\sqrt{6}}$
31. $\pm\frac{1}{a}\{(\sqrt{1+a^2} - 1)(\sqrt{1-a^2} + 1)\}^{\frac{1}{2}}$
32. 6, -1, or $\frac{1}{2}(5 \pm 3\sqrt{-3})$
33. $$\frac{\pm\sqrt{8a^2 + b^2 - 4ac} - b \pm \sqrt{-8a^2 + 2b^2 - 4ac \mp 2b\sqrt{8a^2 + b^2 - 4ac}}}{4a}$$
34. $\pm a\sqrt{\frac{1}{2}(1 \pm \sqrt{5})}$
35. $\frac{2}{3}$, $\pm\frac{4}{3}\sqrt{2}$ or $-\frac{8}{9} \pm \frac{4}{9}\sqrt{-14}$
36. $\pm\frac{3a - 1}{\sqrt{(1 - a)(9a - 1)}}$

37. $\frac{1}{2}(5 \pm \sqrt{17})$ or $\frac{1}{2}(5 \pm \sqrt{-7})$

38. $\frac{1}{2}(9 \pm \sqrt{27 \pm 2\sqrt{-35}})$ or $\frac{1}{2}\left((9 \pm \sqrt{15 \pm 2\sqrt{253}})\right)$

39. $\pm\sqrt{-1}$

40. $18 \pm \frac{5}{2}\{\sqrt{-3} + \sqrt{51 \pm 10\sqrt{-3}}\}$

41. $\frac{1}{24}(7 \pm \sqrt{-47})$

42. $\pm a^{-1}\sqrt{bc}$, $-a$, or $\frac{a}{2}(3 \pm \sqrt{5})$ 43. $-\frac{3}{4}$ or $\frac{-2 \pm \sqrt{-2}}{2}$

44. $\frac{1}{2}(1 \pm \sqrt{-19})$ or $\frac{1}{2}(1 \pm \sqrt{-11})$

45. $\pm\sqrt{3a \pm a\sqrt{a^2 + 2a + 9}}$ where $a = \sqrt{3} - \sqrt{5}$

Exercise LIV.

1. $x = 7$; $y = 2$
2. $x = 13$; $y = 8$
3. $x = 5$ or 4; $y = 4$ or 5
4. $x = 8$ or 7; $y = -7$ or -8
5. $x = \pm 5$ or ± 8; $y = \pm 8$ or ± 5
6. $x = \pm 8$ or $\mp 3\sqrt{-1}$; $y = \pm 3$ or $\pm 8\sqrt{-1}$
7. $x = 12\frac{2}{13}$ or 10; $y = -\frac{4}{13}$ or 4
8. $x = 7$ or $-7\frac{16}{19}$; $y = 4$ or $-5\frac{17}{19}$
9. $x = 11$ or $\frac{13}{43}$; $y = 13\frac{2}{43}$ or -3
10. $x = 3$ or -1; $y = 1$ or -3
11. $x = 2$; $y = 2$
12. $x = 256$ or 1; $y = 1$ or 256
13. $x = 2$ or -46; $y = 3$ or 15
14. $x = 5$ or $-9\frac{1}{4}$; $y = 3$ or $-6\frac{1}{2}$
15. $x = 5$ or $\frac{1}{2}$; $y = 3$ or $-\frac{3}{2}$
16. $x = 2, 4,$ or $3 \mp \sqrt{21}$; $y = 4, 2,$ or $3 \pm \sqrt{21}$
17. $x = 5$ or $1\frac{7}{10}$; $y = 3$ or $-\frac{3}{10}$
18. $x = \pm 7$ or $\mp \frac{11}{2}\sqrt{2}$; $y = \pm 4$ or $\pm \frac{3}{2}\sqrt{2}$
19. $x = \pm 6$; $y = \pm 5$
20. $x = 4$ or 8; $y = 8$ or 4

21. $x = 3$ or 1 or $2 \pm \sqrt{-33}$; $y = 1$ or 3 or $2 \mp \sqrt{-33}$

22. $x = 2$ or 5; $y = 5$ or 2.

23. $x = 3$ or -2 or $\frac{1}{2}\left(1 \pm \sqrt{-31}\right)$; $y = 2$ or -3 or $\frac{1}{2}\left(-1 \pm \sqrt{-31}\right)$

24. $x = 3$ or 4; $y = 4$ or 3.

25. $x = 2$ or 4 or $\frac{1}{6}(-13 \mp \sqrt{377})$; $y = 4$ or 2 or $\frac{1}{6}(-13 \pm \sqrt{377})$

26. $x = \pm 6$; $y = \pm 4$.

27. $x = 3$ or $-1\frac{5}{7}$; $y = 6$ or $4\frac{32}{49}$

28. $x = \frac{m}{2}\left(1 \pm \sqrt{3}\right)$ or $\frac{m}{2}\left(1 \pm \frac{1}{3}\sqrt{3}\right)$; $y = \frac{m}{2}\left(1 \mp \sqrt{3}\right)$ or $\frac{m}{2}\left(1 \mp \frac{1}{3}\sqrt{3}\right)$

29. $x = \pm 3$ or ∓ 8; $y = \pm 5$

30. $x = \pm 2$ or ± 3; $y = \pm 3$ or ± 2

31. $x = \pm 6$; $\mp 4\frac{8}{13}$; $\pm 78\sqrt{3}$; or $\mp 60\sqrt{3}$; $y = \pm 3$ or $\pm 39\sqrt{3}$

32. $x = 5$; $y = 7$.

33. $x = 8$ or $152 \mp 64\sqrt{6}$; $y = 4$ or $40 \mp 16\sqrt{6}$

34. $x = \pm 3$ or $\pm \frac{1}{2}(7 + \sqrt{23})$ or $\pm \frac{1}{2}(2 + \sqrt{22})$; $y = \pm 2$ or $\pm \frac{1}{2}(7 - \sqrt{23})$ or $\pm \frac{1}{2}(\sqrt{22} - 2)$

35. $x = \frac{3}{2}(19 \pm \sqrt{105})$ or $\frac{3}{2}(-13 \pm \sqrt{-87})$; $y = \frac{1}{6}(3 \pm \sqrt{105})$ or $\frac{1}{6}(3 \pm \sqrt{-87}.)$

36. $x = 1$ or $\frac{1}{2}\sqrt[3]{4}$; $y = 0$ or $\frac{1}{2}\sqrt[3]{4}$

37. $x = \pm \sqrt{-1}$ or $\pm \frac{1}{2}\{\sqrt{3 + \sqrt[3]{3}} + \sqrt{\sqrt[3]{3} - 1}\}$; $y = \pm \sqrt{-1}$ or $\pm \frac{1}{2}\{\sqrt{3 + 3\sqrt[3]{9}} + \sqrt{3\sqrt[3]{9} - 1}\}$

38. $x = 4, -2,$ or $1 \pm \frac{1}{11}\sqrt{33}$; $y = 2, -4$ or $-1 \pm \frac{1}{11}\sqrt{33}$.

39. $x = 9, 4,$ or $\dfrac{-13 \pm \sqrt{-51}}{2}$; $y = 4, 9,$ or $\dfrac{-13 \mp \sqrt{-51}}{2}$

40. $x = \pm \frac{1}{2}\sqrt{b}(\sqrt{a+2} \pm \sqrt{a-2})$; $y = \pm \frac{1}{2}\sqrt{b}(\sqrt{a+2} \mp \sqrt{a-2})$, where

$$b = \frac{\sqrt{\dfrac{a-2}{a+2}}}{a-1}$$

41. $x = a$ or $-a^2(a+1)$, $y = -a$ or $\pm a^{\frac{1}{2}} \sqrt{-(a^2+1)}$

42. $x = \pm a \sqrt[4]{\pm 2}$; $y = \pm a \sqrt{(-1 \pm \sqrt{2})}$

43. $x = \pm \frac{1}{17} \sqrt{\{17(a^3 - 9 \pm 3\sqrt{9 - 15a^3 + 2a^6}\}}$;

$y = -\frac{1}{17} \left\{ 6a^3 - 3 \pm \sqrt{9 - 15a^3 + 2a^6} \right\}$

44. $x = \dfrac{m+a}{2}$; $y = \dfrac{m-a}{2}$, where $m = \pm \sqrt{(\pm 2\sqrt{2a^4 + 2b^4} - 3a^2)}$

45. $x = \pm \frac{1}{2} (\sqrt{a^2 - c^2} \pm \sqrt{a^2 + 3c^2})$; $y = \pm \frac{1}{2} (\sqrt{a^2 - c^2} \mp \sqrt{a^2 + 3c^2})$

where $c^2 = \dfrac{a^2 \pm \sqrt{3a^4 - 2b^4}}{2}$

46. $x = \pm \sqrt{14}$ or $\pm \sqrt{\frac{1}{2}(-1 \pm \sqrt{-19})}$;

$y = \pm \sqrt{15}$ or $\pm \sqrt{\frac{1}{2}(1 \pm \sqrt{-19})}$

47. $x = 1$ or $1 \pm \sqrt{-4}$; $y = \pm \sqrt{6}$ or $\pm \sqrt{2 \pm 4\sqrt{-1}}$

48. $x^2 = 1 \pm \sqrt{-97}$, $1 \pm \sqrt{-1}$, $52 \pm \sqrt{2410}$ or $4 \pm \sqrt{10}$

$y^2 = -1 \pm \sqrt{-97}$, $-1 \pm \sqrt{-1}$, $-46 \mp \sqrt{2410}$ or $2 \mp \sqrt{10}$

Exercise LV.

1. 12 and 7 2. 10 and 7 3. 52 and 40 rods
4. 17 and 8 or -8 and -17 5. 12 and 4 6. \$90
7. 16 8. 862 9. 75 ; \$3·20
10. 6 and 4 11. 10 and 14, or 84 and -60
12. $\frac{1}{2}(1 \pm \sqrt{5})$ and $\frac{1}{2}(3 \pm \sqrt{5})$ 13. 4 yds. and 5 yds
14. $\frac{4}{3}$ and $\frac{3}{5}$ 15. 8 16. 3h. 23m
17. 144 miles and 180 miles 18. 16
19. 36 20. Coffee $12\frac{1}{2}$c., Sugar 25c
21. *B*. 30 days, *C*. 36 days 22. $10 \times 10 \times 5$
23. 75 m.; *A*, 15 m. per hour ; *B*, 10 m. per hour
24. $\frac{1}{2}\sqrt{5}$ and $\frac{1}{2}(1 \pm \sqrt{5})$ 25. Bacchus 6h. and Silenus 3h

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Exercise LVI.

1. $1 : d$ 2. $1 : a$ 3. $x + 7 : x + 1$ 4. The former
5. The latter 6. $\dfrac{bc - ad}{c - d}$ 7. ∞ 8. $b : a + b$

Exercise LVII.

1. $\dfrac{bc - ad}{a-b-c+d}$ 4. ± 6 and ± 4 5. 6

8. $\dfrac{2p}{s-d}$ and $\dfrac{2p}{s+d}$ 9. $8 : 7$ 10. \$300 and \$350

13. $3\frac{1}{4}$ 14. $20n^2q : m^2p$ 16. $c^2(a-c)$

Exercise LVIII.

2. $x = \frac{7}{3}y$ 3. $\frac{3}{5}$ 4. $x = \frac{1}{4}\, y\sqrt{y}$

5. $x = \dfrac{36}{15+y}$ 6. $y = 3 + 2x - x^2$ 7. $y = \dfrac{5x^2}{302} + \dfrac{9945}{302x}$

8. $y = b + \dfrac{x^2}{b}$ 10. 143

Exercise LIX.

1. 2883; $n(n+62)$ 2. -1628; $n(6n-206)$

3. 238; $\frac{5}{4}(2m+p) + \frac{3}{4}(2m+p)^2$ 4. $-29\frac{1}{3}$

5. 50; 83; $3n-1$ 6. -77; -132; $8-5n$

7. $13\frac{3}{14}$; $21\frac{11}{14}$; $\frac{5}{14}(5+2n)$ 8. $3 + 10\frac{1}{2} + 18 + 25\frac{1}{2} + 33$

9. $9 - 6 - 21 - 36 - 51 - 66$

10. $-1 + 11\frac{5}{8} + 24\frac{1}{4} + 36\frac{7}{8} + 49\frac{1}{2} + 62\frac{1}{8} + 74\frac{3}{4} + 87\frac{3}{8} + 100$

11. 2701 12. $2n-1$

14. at^2 15. $39a$; $a(2t-1)$

16. $\pm 14, \pm 10, \pm 6, \pm 2$ 17. $\pm 14, \pm 10, \pm 6, \pm 2$

18. 1, 3, 5, 7, 9, or 9, 7, 5, 3, 1 19. \$1·00$\frac{61}{120}$ 22. 11

23. 2,5,8,11,14,17,20,23,26,29,32,35 24. 11, 10, 9, 8, 7, 6, 5

25. $b - c + 2ct$ 28. $\dfrac{n}{4}(27-n)$ 29. $\pm 1, \pm 3, \pm 5$

30. 2, 4, 6 and 8, or 8, 6, 4 and 2

Exercise LX.

1. 729; 1092 2. 256; 511 3. $18\frac{2}{7}$; $36\frac{2}{7}$

4. -6144; -4095 5. $-12\frac{153}{256}$; $-5\frac{1}{256}$ 6. $-\frac{15}{64}$; $19\frac{59}{64}$

7. $-\frac{4}{5}$ 8. $1\frac{1}{5}$ 9. $4\frac{2}{3}$ 10. $42\frac{2}{3}$ 11. $\frac{623}{999}$

12. $\frac{7}{9}$ 13. $\frac{967}{990}$ 14. $\frac{8537}{9900}$ 15. $\frac{1}{2}(3^n - 1)$

16. $\frac{10}{7}\{1-(-\frac{2}{5})^n\}$ 17. $62(1+\sqrt{2})$ 18. $\dfrac{a^{qn+p}-a^p}{a^q-1}$

19. $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}$

20. $2+6+18+54+162+486+1458+4374+13122$

21. $9+3+1+\frac{1}{3}+\frac{1}{9}$ 22. 4, 24, 144 and 864

23. 5, 10, 20 and 40 or − 15, 30, − 60 and 120

24. \$180, \$90 and \$45, or \$375, − \$300 and \$240

25. 2, 4, 8, 12 and 16 29. 5, 10, and 20, or $46\frac{2}{3}$, $-23\frac{1}{3}$ and $11\frac{2}{3}$

30. 248

Exercise LXI.

1. (i) $\frac{1}{13}, \frac{1}{11}, \frac{1}{9}, \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, 1, -1, -\frac{1}{3}$
 (ii) $\frac{1}{30}, \frac{1}{26}, \frac{1}{22}, \frac{1}{18}, \frac{1}{14}, \frac{1}{10}, \frac{1}{6}, \frac{1}{2}, -\frac{1}{2}$
 (iii) $-\frac{1}{4}, -\frac{1}{2}, \infty\ \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}$
 (iv) $-\frac{14}{23}, -\frac{14}{13}, -2, 14, 1\frac{5}{9}, \frac{14}{17}, \frac{14}{25}, \frac{14}{33}$, and $\frac{14}{41}$
 (v) $\frac{5}{32}, \frac{1}{6}, \frac{5}{18}, \frac{5}{11}, 1\frac{1}{4}, -1\frac{2}{3}, -\frac{1}{2}, -\frac{5}{17}, -\frac{5}{24}$
 (vi) $-\frac{1}{8}, -\frac{1}{6}, -\frac{1}{4}, -\frac{1}{2}, \infty, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ and $\frac{1}{8}$

2. (i) $2+2\frac{2}{11}+2\frac{2}{5}+2\frac{2}{3}+3$
 (ii) $5+5\frac{5}{13}+5\frac{5}{6}+6\frac{4}{11}+7$
 (iii) $11+6\frac{3}{5}+4\frac{5}{7}+3\frac{2}{3}+3$
 (iv) $2\frac{1}{4}+2\frac{46}{109}+2\frac{94}{151}+2\frac{238}{277}+3\frac{1}{7}$
 (v) $6-2-\frac{6}{7}-\frac{6}{11}-\frac{2}{5}$

3. $\frac{5}{14}$; $\frac{5}{32}$, and $\dfrac{5}{3n-1}$ 4. $1\frac{5}{8}$; $1\frac{1}{12}$ and $\dfrac{13}{n+2}$

5. $\frac{1}{16}$ and $\frac{1}{24}$ 6. 2 and $1\frac{1}{3}$

7. $\dfrac{ab}{7a-6b}$; $\dfrac{ab}{b(2-n)+a(n-1)}$ 8. $\dfrac{1}{m}$

9. $6\frac{1}{2}$; 6; $5\frac{7}{13}$ 10. $5\frac{1}{12}$; 5; $4\frac{56}{61}$ 13. Half of the middle term

14. 18 and 2 15. 14 or $\frac{2}{7}$ 16. 20 and 10

17. $20\frac{1}{4}$ and 4

Exercise LXII.

1. 720 2. (i) 1680; (ii) 20160; (iii) 40320

3. 360360 4. 136 yrs. 222 days 5. $n=6$

6. Loss = \$25465000 when the money is not paid till the end of the period.

Loss = \$22536215 when the \$5000 is paid down and placed at interest for the whole period

7. $n=6$ 8. 3634108800; 39916800; 1680; 1729728

9. 2520; 778377600; 420 10. $n=12$

Exercise LXIII.

1. 120, 252 ; 45 2. 3003 ; 6435 ; 435 3. 792. 4. $n = 9$

5. 439824 ; 52360 6. 30164400

7. 362880 or 181440 according as B, A, C and C, A, B are regarded as giving A different or the the same neighbours

8. $n = 7$ 9. 15 and 6

10. $\lfloor n-1$ or $\frac{1}{2} \lfloor n-1$ (See Ans. 7) 11. 637 12. 511

Exercise LXIV.

1. $1 - 3x + 6x^2 - 10x^3 + 15x^4 - \&c$
2. $1 - 2x + 3x^2 - 4x^3 + 5x^4 - \&c$
3. $1 + 2x + 4x^2 + 8x^3 + 16x^4 + \&c$
4. $1 + \frac{5}{2} x + \frac{15}{4} x^2 + \frac{35}{8} x^3 + \frac{35}{8} x^4 + \&c$
5. $1 - 6x + 27x^2 - 108x^3 + 405x^4 - \&c$
6. $1 + 10x + 60x^2 + 280x^3 + 1120x^4 + \&c$
7. $1 + 4x + 10x^2 + 20x^3 + 35a^4 + \&c$
8. $1 - 2x - 2x^2 - 4x^3 - 10x^4 - \&c$
9. $1 - \frac{2}{3} x + \frac{5}{9} x^2 - \frac{40}{81} x^3 + \frac{110}{243} x^4 - \&c$
10. $1 - \frac{3}{5}x - \frac{9}{200}x^2 - \frac{27}{2000}x^3 - \frac{891}{160000}x^4 - \&c$
11. $1 + \frac{2}{9}x - \frac{4}{81}x^2 + \frac{40}{2187}x^3 - \frac{160}{19683}x^4 + \&c$
12. $1 + \frac{4}{5}x + \frac{18}{25}x^2 + \frac{84}{125}x^3 + \frac{399}{625}x^4 + \&c$
13. $a^{-2} + 3a^{-3}x^2 + 6a^{-4}x^4 + 10a^{-5}x^6 + 15a^{-6}x^8 + \&c$
14. $a^{-2} - a^{-4}x^3 + a^{-6}x^6 - a^{-8}x^9 + a^{-10}x^{12} - \&c$
15. $a^{-1} + 2a^{-\frac{3}{2}}x^{\frac{1}{3}} + 3a^{-2}x^{\frac{2}{3}} + 4a^{-\frac{5}{2}}x + 5a^{-3}x^{\frac{4}{3}} + \&c$
16. $a^{\frac{8}{3}} - \frac{2}{3} a^{-\frac{4}{3}}x^3 - \frac{1}{9} a^{-\frac{16}{3}}x^6 - \frac{4}{81} a^{-\frac{28}{3}}x^9 - \frac{7}{243} a^{-\frac{40}{3}}x^{12} - \&c$
17. $a^{-\frac{4}{3}} - 4a^{-\frac{16}{3}}x^{-2} + 10a^{-\frac{22}{3}}x^4 - 20a^{-\frac{31}{3}}x^6 + 35a^{-\frac{40}{3}}x^{-8} - \&c$
18. $a^{-\frac{1}{15}} + \frac{1}{3} a^{-\frac{4}{15}}x^{-\frac{1}{5}} + \frac{2}{9} a^{-\frac{7}{15}}x^{-\frac{2}{5}} + \frac{14}{81} a^{-\frac{2}{3}}x^{\frac{3}{5}} + \frac{77}{486} a^{-\frac{13}{15}}x^{-\frac{4}{5}} + \&c$
19. $a^{-\frac{4}{3}}m^{-\frac{2}{3}} + \frac{2}{3} a^{-\frac{10}{3}}m^{-\frac{5}{3}}x^{\frac{1}{2}} + \frac{5}{9} a^{-\frac{16}{3}}m^{-\frac{8}{3}}x + \frac{40}{81} a^{-\frac{22}{3}}m^{-\frac{11}{3}}x^{\frac{3}{2}} +$
$\frac{110}{243}a^{-\frac{28}{3}}m^{-\frac{14}{3}}x^2$
20. $a^{\frac{2}{5}} + \frac{2}{5} a^{-\frac{3}{5}}x^{-3} - \frac{3}{25} a^{-\frac{8}{5}}x^6 + \frac{8}{125} a^{-\frac{13}{5}}x^9 - \frac{26}{625} a^{-\frac{18}{5}}x^{12} + \&c$
21. $a^{-\frac{1}{2}} + \frac{1}{2} a^{-\frac{3}{2}} bx + \frac{3}{8} a^{-\frac{5}{2}} b^2x^2 + \frac{5}{16} a^{-\frac{7}{2}}b^3x^3 + \frac{35}{128} a^{-\frac{9}{2}} b^4x^4 + \&c$

EXERCISE LXV.

1. $\dfrac{3.4.5\ldots.(2+r)}{\underline{|r}}\,x^r$ and $21x^5$

2. $(-1)^r\left(\dfrac{4.5.6\ldots.(3+r)}{\underline{|r}}\right)x^r$ and $-56x^5$

3. $(-1)^r\left(\dfrac{2.5.8\ldots.(3r-1)}{\underline{|r}\times 3^r}\right)x^r$ and $-\dfrac{308}{729}x^5$

4. $(-1)^r\left(\dfrac{4.1.-2\ \ldots.\ (7-3r)}{\underline{|r}\times 3^r}\right)x^2$ and $\dfrac{8}{729}x^5$

5. $(-1)^r\left(\dfrac{7.9.11\ldots.(5+2r)}{\underline{|r}\times 2^r}\right)x^r$ and $-\dfrac{9009}{256}x^5$

6. $(-1)^r\left(\dfrac{8.11\ldots.(5+3r)}{\underline{|r}\times 3^r}\right)x^r$ and $-\dfrac{10472}{729}x^5$

7. $a^{-(r+1)}x^r$ and $a^{-6}x^5$

8. $(-1)^r\left(\dfrac{6.1.4\ldots.(5r-11)}{\underline{|r}\times 10^r}\right)a^{\frac{6}{5}-r}x^r$ and $-\dfrac{63}{250000}a^{-\frac{19}{5}}x^5$

9. $(r+1)\,2^r\,x^r$ and $160x^5$

10. $(-1)^r\left(\dfrac{5.7\ldots.(3+2r)}{\underline{|r}\times 3^r}\right)x^{2r}$ and $\dfrac{385}{216}x^8$

11. $(-1)^r\times\dfrac{2.7\ldots.(5r-3)}{\underline{|r}\times 5^r}\,a^{2r+\frac{4}{3}}\,x^{-\frac{2}{3}r}$; and $\dfrac{119}{625}a^{\frac{28}{3}}x^{-\frac{8}{3}}$

12. $(r+1)\,a^{\frac{r+2}{2}}\,x^{-\frac{r}{2}}$; and $5a^3x^{-2}$.

13. 1024 14. 128 15. 0 16. 4096

17. The 4th term = 32 18. The 4th = the 5th = $4\frac{3}{8}$.

19. 13th term 20. 9th = 10th = $\dfrac{19702683}{390625}$

EXERCISE LXVI.

1. $x<5$ 2. $x>12$ 3. $x<3$

4. $x>-10$ 6. $x>a$ and $<b$ 9. $x=5$

Exercise LXVII.

1. n 2. $\frac{3a}{2}$ 3. $\frac{1}{2}$ 4. $1\frac{1}{2}$ 5. $-2\frac{1}{2}$

6. $\frac{x^2+b}{x+b^2}$ 7. $\frac{a}{b}$ 8. ∞ 9. $\frac{3a}{a-6}$

Exercise LXVIII.

1. $x = 2, y = 1$ 2. $\begin{cases} x = 10, 23, 36, 49, \&c \\ y = 3, 8, 13, 18, \&c \end{cases}$

3. $\begin{cases} x = 26, 19, 12 \text{ or } 5 \\ y = 1, 3, 5 \text{ or } 7 \end{cases}$ 4. $x = 3$ and $y = 1$

5. $\begin{cases} x = 4, 21, 38, 55, \&c \\ y = 2, 11, 20, 29, \&c \end{cases}$ 6. $x = 2$ and $y = 3$

7. $\begin{cases} x = 2, 43, 84, 125, \&c \\ y = 1, 13, 25, 37, \&c \end{cases}$ 8. $x = 5$ and $y = 4$

9. $\begin{cases} x = 12, 55, 98, \&c \\ y = 6, 28, 50, \&c \end{cases}$ 10. $x = 11$ and $y = 4$

11. $\begin{cases} x = 5, 165, 325, \&c \\ y = 1, 100, 199, \&c \end{cases}$ 12. $\begin{cases} x = 2, 6, 10, 14, \&c \\ y = 3, 20, 37, 54, \&c \end{cases}$

13. $x = 2, y = 3, z = 4$ 14. $x = 11, y = 3, z = 2$

15. 45 16. 54

17. He pays 8 guineas and receives back 7 half-crowns

18. $x = 2n$ and $y = n^2 - 1$ where n may be assumed at pleasure = any integral number; and it will be found that $x^2 + y^2$ is a square

19. $x = \frac{n^2+1}{2^n} \cdot y$ where n and y may be assumed at pleasure and it will be found that $x^2 - y^2$ is a square

20. 98. 21. 109.

22. No two fractions with denominators 10 and 15 added together will make $\frac{29}{60}$. Prove this.

23. The problem is impossible Prove this.

24. 3, 6, 9, 12 or 15 £5 notes; 81, 62, 43, 24 or 5 £1 notes; 16, 32, 48, 64 or 80 crown-pieces.

25. 22 and 3; 16 and 9; 10 and 15; or 4 and 21

26. 3, 15 and 6; 7, 8 and 9; or 11, 1 and 12

28. $2^n \times (2^{n+1} - 1)$ where n may be assumed = to any integral number.

29. 417 30. 1 at \$50, 9 at \$30, and 90 at \$2.

MISCELLANEOUS EXERCISES.

1. $\dfrac{17 - 21a}{36}$ 3. $a + b$ 4. $\dfrac{a}{6}$ 5. $x = 1,\ y = 5,\ z = 9$

6. $3\sqrt[3]{3}$ 7. $\dfrac{\pm 1 \pm \sqrt{-1}}{\sqrt{2}}$ 11. $x^n + 1 + x^{-n}$; $x^{\frac{1}{3}} - a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}$

12. $7x^2 - 3xy + 4y^2$ 13. x^{m+n+p}; abc

14. $4x^4 + y^2 + \frac{1}{4}x^{-4}y^4$; $x^4 + b^4 + 2b^2x^2 - a^2x^2$; $x^{m+n} + x^my^q + x^ny^p + y^{p+q}$

15. $\frac{1}{31}(69 - 17\sqrt{15})$

16. $\dfrac{12x^2 + 1}{12x^3 + 6x}$ 17. $\dfrac{4x^2 + 2x + 1}{16x^4 - 1}$ 18. $x = -\frac{1}{2}a$; (II) x has no possible roots (III) $x = 12 \pm \sqrt{269}$.

20. $x = \dfrac{2abc}{ac + bc - ab}$; $y = \dfrac{2abc}{bc - ac + ab}$; $z = \dfrac{2abc}{ab + ac - bc}$

21. 1. 23. $(a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2)$; $(a^2 + ab\sqrt{3} + b^2)(a^2 - ab\sqrt{3} + b^2)$

24. $\dfrac{x}{y}$. 25. $G.\,C.\,M. = x - 4y$; $l.\,c.\,m. = x^4 + 4x^3y - 27x^2y^2 - 34xy^3 + 56y^4$

26. $S_n = na$ or $S_n = 0$ or a according as $r = +1$ or -1, and n an even or odd number

27. $4\cdot 9s$ per day 29. (I) $2059\frac{241}{256}$; (II) $5\frac{71}{81} + 4\frac{76}{81} + 4 +$ &c (III) $9, 6, 4, 2\frac{2}{3}$, &c

30. 110×50 31. (I) $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$;

(II) $7x^5 - 14x^3 + 7x^2 + 33x - 32 - \dfrac{59x^2 - 100x + 23}{x^3 + 2x - 1}$

(III) $x^{m-1} + x^{m-3} + x^{m-5} + x^{m-7} +$ &c., rth term $= x^{m-2r+}$

32. (I) $5 + 2\sqrt{3}$; (II) $\sqrt{2x+1} + \sqrt{2x-1}$ 33. $15a^{-28}x^{-16}$

34. 1184040 35. $x^2 - 2 + 3x^{-2}$ 36. 1 or $\frac{1}{2}(-1 \pm \sqrt{-3})$

37. (I) $x = \dfrac{a^2}{a - b}$; $y = \dfrac{b^2}{b - a}$;

(II) $x = 0, 10, 4$ or -2; $y = 0, 10 - 2$ or 4

38. 3 and $3\frac{1}{3}$ 39. $\frac{1}{2} + \frac{6}{17} + \frac{3}{11} + \frac{2}{9} + \frac{3}{16} + \frac{6}{37} + \frac{1}{7}$

40. An identity

41. 0 or $\frac{1}{8}(1 \pm 3\sqrt{-7})$ 42. $ab + bc + ac$

43. $\frac{13}{12}x^2 + \frac{1}{3}xy - \frac{17}{30}y^2 + (p - m)x + (n - q)y$, or $\frac{1}{12}x^2 - \frac{5}{4}xy - \frac{1}{30}y^2 - (m + p)x + (n + q)y$

44. $\dfrac{x+7}{x-5}$ 45. (I) $x = a$ or b; (II) $x = \frac{2}{3}$ 46. $x = \pm 3$ or $\pm \frac{1}{3}\sqrt{3}$; $y = \pm 4$ or $\pm \frac{10}{3}\sqrt{3}$; $z = \pm 2$ or $\mp \frac{8}{3}\sqrt{3}$ 47. $a^2(b+1)^2$

48. $x^{3p} - 2x^p + 3x^{-3p}$ 49. $107\frac{3}{7}$; $\dfrac{6561x^{14} - 256x^{-2}y^8}{81x^2 + 54y}$; $60\frac{3}{4}$

50. Any series having $r = 2$ 51. 1 52. $\dfrac{x^2+x+1}{x^2-x+1}$

53. $x = 3a - b$ or $3b - a$ 54. $x = 15$; $y = 20$

55. $x^4 + 4x + 3$; $x^8 - 4xy^{\frac{9}{2}} + 3y^6$

56. $x = \frac{1}{2}(a^3 + 3b^3)^{\frac{1}{3}}\left(1 \pm \sqrt{\dfrac{a^3 - b^3}{a^3 + 3b^3}}\right)$

$y = \frac{1}{2}(a^3 + 3b^3)^{\frac{1}{3}}\left(1 \mp \sqrt{\dfrac{a^3 - b^3}{a^3 + 3b^3}}\right)$

57. 7 58. $4a^{\frac{1}{2}}b^{\frac{3}{2}}$ 59. $(4a^4 - \frac{13}{2}a^2)$; $(12a^8 - a)$

60. $x = \dfrac{(a+b)^2}{2(a-b)}$; $y = \frac{1}{2}(a+b)$ 61. $\dfrac{30x-23}{13x-10}$; $\dfrac{x(x^2+1)^2}{x^6-2x^4+2x^2-1}$

62. $\dfrac{x^2 - 9x + 24}{x^3 + 5x^2 - 29x - 105}$ 63. $\dfrac{x}{y} - \dfrac{1}{2} + \dfrac{y}{x}$; $x^2 - x + \frac{1}{4}$

64. (I) $x^{2m} - 3x^m y^n + 2y^{2n}$ (II) $x^{2m^2} - a^2 x^{2m} + 2abx^m + b^2$

65. $4x^3 + 8x^2 + 16x + 32$; $5a^2b^3 - 3ab^4$ 67. 3, 4, 5, 6, or 7

68. 30 71. (I) $a^4 - 5a^3 + 25a^2 - 138a + 790 - \dfrac{4507a - 3166}{a^2 + 5a - 4}$

(II) $x^4 + 2x^2 + 3 + 2x^{-2} + x^{-4}$.

72. $x - a$ 73. $\dfrac{36x^2 + 18x + 29}{16x^4 - 81}$

74. $\frac{1}{16}x^4 + \frac{4}{81}y^4$; $64x^{\frac{3}{2}} - 16xy^{\frac{1}{2}} + 36x^{\frac{1}{2}}y - 729y^{\frac{3}{2}}$

75. $x = 3\sqrt{3}$; $y = 2\sqrt{2}$

79. $v = 1 - \dfrac{17x}{13y} + \dfrac{4}{13}xy^2$ 80. $x = \pm 1$; $x = \dfrac{b(b \pm \sqrt{b^2 + 4a^2 - 4ab})}{2a(a-b)}$

81. $a^{2^n} - b^{2^n}$ 82. $x^2 - (a+b)x - c$ 83. $ax^3 + bx + cx^{-1}$

84. $\dfrac{x^2 - ab}{x^2 + ab}$ 85. $x^2 + px + p^2$ 86. $\frac{25}{3}(x^3 - 5x^2 - 26x + 120)$

87. $\dfrac{b-1}{a+1}$ 88. $\dfrac{x^6 - 1}{x^6 + 1}$ 90. By A in 2, B in 3, and C in 4 hours 91. $x^2 + x - 3$

92. $apqx^3 + (aq^2 + bpq - ap^2)x^2 - (apq + bp^2 - bq^2)x - bpq$
II) $x^3y - xy^3$

93. (I) $11a$; (II) $\pm\sqrt{7}$; (III) 5 or -12; (IV) $x = 1$ or $\pm\sqrt{15}$, $y = 5$ or $\frac{1}{3}$

94. $x^4 - x^3 - 7x^2 - 11x + 42 = 0$ 95. m 96. $(a^4 + a^2b^2\sqrt{2} + b^4)(a^4 - a^2 b^2 \sqrt{2} + b^4)$

98. $1h\ 5\frac{5}{11}m$ 99. $x = \dfrac{a(cd - e - bc)}{bc - ad}$; $y = \dfrac{b\ (cd - e - ad)}{bc - ad.}$; Problem indeterminate.

100. $\frac{1}{2}(a + b)$ 101. $5x^3 + 10x^2 + 5x - 23 - \dfrac{61x - 70}{x^2 - 2x + 3}$ or $5x^3 + 10x^2 + 5x - 23 - 61x^{-1} - 52x^{-2} + 79x^{-3} +$ &c

102. $x - y$; if $y = 1$ the *G. C. M.* is $x^2 + 4x - 5$

104. $(a^2 + am\sqrt{2} + m^2)(a^2 - am\sqrt{2} + m^2)(a^4 + a^2 m^2 \sqrt{3} + m^4)(m^4 - a^2 m^2 \sqrt{3} + m^4)$

105. 1 106. 3

114. 0 115. $7x^2 + 7xy + 7y^2$ 116. $2x^2 + x - 1$

117. $(2x - 1)(x + 1)(3x + 2)(3x - 2)$ and $(2x - 1)(x + 1)(2x + 1)(2x - 1)$

118. $\dfrac{1 + x + x^2}{1 - x - x^4 + x^5}$ 119. An indeterminate equation; an identity

123. 11, 9, 7, 5, &c 125. $3 - 2 + \frac{4}{3} - \frac{8}{9} + \frac{16}{27} -$ &c

128. $\dfrac{2618}{6561}x^{-12}$; $-\dfrac{391391}{1594323}x^{-18}$; $(-1)^r \times \dfrac{2.5.8\ldots(3r-1)}{\underline{|r} \times 3^r}x^{-2r}$

129. $x^6 - 6x^5 + 6x^4 + 30x^3 - 51x^2 - 24x + 44 = 0$

130. $\frac{1}{2}(-3 \pm \sqrt{5})$ 131. $\dfrac{4bc - ad}{d - 4c}$

133. $x = 2$, $y = 3$, $z = 4$ 134. $\dfrac{n + 1}{x^n}$

135. 21 and 24 136. $1 \pm \sqrt{19}$ 137. $x = 10$, $y = 8$

140. $\frac{1}{2}\{\pm\sqrt{4nab + (a - b)^2} - (a + b)\}$ 141. $\frac{1}{4}$ or $\frac{1}{9}$ 142. $\frac{2}{3}\sqrt{3}$

143. $x = \pm\dfrac{a^2}{\sqrt{a^2 + b^2 + c^2}}$; $y = \pm\dfrac{b^2}{\sqrt{a^2 + b^2 + c^2}}$; $z = \pm\dfrac{c^2}{\sqrt{a^2 + b^2 + c^2}}$

145. $b^2 - 1$. 146. $\dfrac{x^3y^3}{27} + 27$. 147. $a^2 + b^2 + c^2 + d^2$.

148. $\pm\{a(x + 2) - 1\}$. 149. $a + b - c$.

150. $4x - \dfrac{3x^2 - 4x - 1}{x^3 + 2x - 1}$; $4x - 3x^{-1} + 4x^{-2} + 7x^{-3} - 11x^{-4} -$ &c

151. $1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \&c$

152. $\dfrac{a^4 + 2a^2b^2 + b^4}{a^4 - 2a^2b^2 + b^4}$ 153. 1 154. $\dfrac{1}{(x-1)(x-2)(x-3)}$

156. I. They must have a common measure; II. The coefficients of x must be = but of opposite signs, and the coefficients of x^2 must be =, and also those of x^0 must be =

157. $2\frac{3}{5}$ 158. $1 \pm 2\sqrt{3}$ 159. $\pm \frac{1}{4}\sqrt{3}$, or 0

160. $x = \dfrac{1+P}{1-P} \pm \dfrac{2}{(n-1)(1-P)}\sqrt{(Pn-1)(n-P)}$

161. *A. M.* = $1\frac{1}{24}$; *G. M.* = 1; *H. M.* = $\frac{24}{25}$ 163. 0; 217

164. $\dfrac{25}{3} - \dfrac{3^{n-1}}{5^{n-2}}$. 165. $\frac{5}{7}\{1 - (-\frac{2}{5})^n\}$; $\frac{5}{7}(-\frac{2}{5})^n$.

167. $x^4 + x^2y^2 + y^4$ 168. 43 169. $2a - 3b$. 171. 5.

172. a or $-\dfrac{b(a+b)}{2a+b}$ 173. $\dfrac{a}{a+ab+1}$ 175. $(a-b)^2 + c^2$

176. $x + 1$ 177. $\dfrac{14x - 4x^2 + 14}{(x+5)(x^4-1)}$ 178. $\dfrac{2(a^2-b^2)}{a^2+b^2}$

181. $\pm\sqrt{b(2a-b)}$ 182. 64 or $\frac{27}{9}\sqrt[5]{7857}$ 183. $\pm\sqrt{ab}$

184. 5 or $6\frac{11}{4\times 3}$ 185. 42

186. A's rate 1st round is 10 miles per hour, 2nd round 12 miles per hour; B's rate, 12 miles per hour first round, and 10 miles per hour second. Neither wins

189. $x^6 + x^4y^2 - x^2y^4 - y^6$ 190. b^2 191. $ax^2 + 2cyx + by^2$

192. $x^n + 1$ 193. $x + 4$ 194. $12abc$

195. $(a+b+c)(x+y+z)$ 196. $x^6 - 12x^4y^2 + 48x^2y^4 - 64y^6$; $a^2 + b^2$ 197. $2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$; $x^2 + 1 + x^{-2}$

198. $\frac{1}{3}x^2y^{-2} - \frac{1}{2}xy^{-1} + \frac{2}{3} - \frac{5}{6}x^{-1}y + x^{-2}y^2$; $x^2 - 2x + 3$

199. $8x^{\frac{2}{3}} + 4x^{\frac{1}{3}}y + 2y^2$; $x^2 + (1-p)ax + a^2$

200. $x - 4$; $x^2 - 1$; $x^{2p} - a^{2p}$ where p is the *G. C. M.* of m and n

201. $ax^5 - 2a^2x^4 - a^3x^3 + 2a^4x^2$; $x^4 - x^2y^2 - a^2x^2 + a^2y^2$

202. $1 + \dfrac{2d}{a+b+c+d}$ 203. $\dfrac{x+y+z}{x-y+z}$ 204. $\dfrac{3a}{a+b}$

205. $\dfrac{a^4}{2(a^2+x^2)}$ 206. 2 207. $x^2 - 2x - 2$; $2x^{2n} - \frac{1}{3}x^{3n}$

$\dfrac{a}{b} - \dfrac{b}{c} - \dfrac{c}{a}$

209. $\frac{am}{n}$; 6 or $\frac{1}{2}$ 210. $2 \cdot 14$ or $-0 \cdot 49$; $3a - b$ or $3b - a$

211. $x = \frac{ac + b}{1 + a^2}$ $y = \frac{c - ab}{1 + a^2}$; $x = 4$ or 3, $y = 3$ or 4

212. $x = \pm \frac{bc}{a}$; $y = \pm \frac{ac}{b}$, $z = \pm \frac{ab}{c}$ 213. A, 15; B, 21; C, 24

214. 117; $\frac{1}{2}\{n(n+1) + 4 - (\frac{2}{3})^n\}$; $3\sqrt{2} + 2\sqrt{3}$ 216. $5 \pm 2\sqrt{7}$

217. 0, 1, $\frac{1}{25}(4 \pm \frac{1}{2}\sqrt{754})$.

218. $2n(4n+1) + \frac{1}{3}(1 - 16^n)$; $(2n+1)(4n+1) + \frac{1}{3}\{1 - (-2)^{4n+1}\}$; $(4n+3)(2n+1) + \frac{1}{3}(1 - 4^{2n+1})$; $2(n+1)(4n+3) + \frac{1}{3}\{1 - (-2)^{4n+3}\}$

219. 72 221. 90 miles; \$2·70

222. $x = 0$, or $\frac{1}{2}\left(\sqrt{10\sqrt{5} - 70}\right)$ or $\frac{1}{2}\left(\sqrt{10\sqrt{29} - 46}\right)$ or $\pm 3\sqrt{-1}$

224. I $\frac{4m}{n}(m \pm \sqrt{m^2 - n^2})$ and $\frac{4m}{n}(m \mp \sqrt{m^2 - n^2})$

II $4(m \pm \sqrt{m^2 - mn})$ and $4(m \mp \sqrt{m^2 - mn})$

225. Ages at first trial = 11 and 15

Throws at first trial = 66 and 90 feet,

And at second trial = 74 and 96 feet.

THE END.